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ASYMMETRY IN REAL FUNCTIONS THEORY

JACEK MAREK JĘDRZEJEWSKI

Abstract

Since the beginning of the XX century many authors considered characterizations of local properties for real functions of a real variable which have been defined as global properties. We present a short survey of local properties of the well known global ones and consider of how small/big the set of asymmetrical behaviour of a function must be.

1. Introduction

We shall consider only real functions defined in an open interval. When we use topological terminology, then it is applied in the sense of natural topology in the set of real numbers (or in its subsets).

Limit numbers of a real function defined in subsets of $\mathbb{R}$ have been considered in many articles by many mathematicians. Starting from the classical result of W. H. Young [20] concerning asymmetry of functions through problems of usual limit numbers, J. M. Jędrzejewski and W. Wilczyński [12], approximate limit numbers discussed by M. Kulbacka [14], L. Belowska [1], W. Wilczyński [18] and others, problems of qualitative limit numbers (W. Wilczyński [19]) $\mathcal{B}$-limit numbers (J. M. Jędrzejewski [7], [8], J. M. Jędrzejewski together with W. Wilczyński [13]) one can come up to a big monograph on local systems by B. S. Thomson [17].

The first part of our considerations deals with the asymmetry of functions with respect to limit numbers of different kinds.

Some properties of functions (continuity, Darboux condition and others) can be characterized globally and locally. For many of those properties we have theorems which say that a function has this global property if and only if it has its adequate local property. The second part of the article deals with some of such properties.

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The last part of the paper is devoted to results obtained by T. Świątkowski in view of general approach to limit numbers considered originally by B. S. Thomson and me.

2. Asymmetry of Sets of Limit Numbers

2.1. **Limit Numbers of a Real Function.** We shall start with the classical problem called Rome's Theorem. The theorem was probably the first one which dealt with arbitrary function. Let us remind necessary definitions and properties.

Definition 1. (W. H. Young [20]) *Let a real function f be defined in an open interval (a, b). Then a number g (or $+\infty$ or $-\infty$) is called the limit number of f at a point x_0 from (a, b) if there exists a sequence $(t_n)_{n=1}^{\infty}$ such that*

(1) *$t_n \neq x_0$, for each positive integer n,*
(2) *$\lim_{n\to\infty} t_n = x_0$,*
(3) *$\lim_{n\to\infty} f(t_n) = g$.*

If the inequality $t_n \neq x_0$ is replaced by $t_n > x_0$, then such a limit number is called the right limit number of f at x_0.

If the inequality $t_n \neq x_0$ is replaced by $t_n < x_0$, then such a limit number is called the left limit number of f at x_0.

- By $L^+(f, x_0)$ we denote the set of all right limit numbers of f at x_0.
- By $L^-(f, x_0)$ we denote the set of all left limit numbers of f at x_0.
- By $L(f, x_0)$ we denote the set of all limit numbers of f at x_0.

Let us remark that limit numbers can be equivalently defined in the following way:

Theorem 1. *Let a real function f be defined in an open interval (a, b). Then a number g (or $+\infty$ or $-\infty$) is a limit number of f at a point x_0 from (a, b) if and only if the set*

$$\left\{x \in (a, b) \,:\, f^{-1}(U_g) \cap [(x_0 - \varepsilon, x_0 + \varepsilon) \setminus \{x_0\}]\right\}$$

is non-empty for each positive ε and each neighbourhood U_g of the point g.

It is obvious that:

Theorem 2. *The sets $L^-(f, x_0)$, $L^+(f, x_0)$ and $L(f, x_0)$ are non-empty and closed, moreover*

$$L(f, x_0) = L^-(f, x_0) \cup L^+(f, x_0)$$

for any function $f\colon (a, b) \longrightarrow \mathbb{R}$ and any $x \in (a, b)$.

The main theorem which was announced in Rome at the congress of mathematicians is stated as follows:

Theorem 3. Rome's Theorem on Asymmetry (W. H. Young, 1906) *For any function* $f\colon (a,b) \longrightarrow \mathbb{R}$ *the set*

$$\big\{x \in (a,b) \,:\, L^-(f,x) \neq L^+(f,x)\big\}$$

is at most countable.

Quite similarly one can say that:

Theorem 4. *For any function* $f\colon (a,b) \longrightarrow \mathbb{R}$ *the set*

$$\{x \in (a,b) \,:\, f(x_0) \notin L(f,x)\}$$

is at most countable.

Let us remark that for each countable set E in $\mathbb{R}$ there exists a function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ for which

$$E = \big\{x \in (a,b) \,:\, L^-(f,x) \neq L^+(f,x)\big\}\,.$$

It is quite obvious if the set E is finite; if it is infinite it is possible to define a monotone function, which fulfils the required condition. We shall construct such a function.

Example 1. Monotone function with infinite set of asymmetry.

Let $E = (x_n)_{n=1}^{\infty}$ and the sequence of positive numbers $(\alpha_n)_{n=1}^{\infty}$ be such that the series $\sum\limits_{n=1}^{\infty} \alpha_n$ is convergent. The function

$$f(x) = \sum_{\{n\,:\,x_n<x\}} \alpha_n$$

fulfils all the required properties.

2.2. **Qualitative Limit Numbers.** Following the way as in Theorem 1. one can define other kinds of limit numbers as qualitative (W. Wilczyński [19]) or approximative limit numbers (L. Belowska [1], M. Kulbacka [14], J. Jaskuła [5] and W. Wilczyński [18]) when we define limit numbers using the above mentioned property.

Definition 2. *A number* g *or* $+\infty$ *or* $-\infty$ *is called the qualitative limit number of a function* f *at a point* x_0 *if the set*

$$\big\{x \in (a,b) \,:\, f^{-1}(U_g) \cap (x_0-\varepsilon, x_0+\varepsilon)\big\}$$

is of the second category for each positive ε *and arbitrary neighbourhood* U_g *of the point* g*.*

Definition 3. *If the set*

$$\{x \in (a,b) \,:\, f^{-1}(U(g)) \cap (x_0 - \varepsilon, x_0)\}$$

is of the second second category for each positive ε, then g is called the left qualitative limit number of a function f at the point x_0.

Similarly, g is called the right qualitative limit number of a function f at a point x_0 if the set

$$\{x \in (a,b) \,:\, f^{-1}(U(g)) \cap (x_0, x_0 + \varepsilon)\}$$

is of the second category for each positive ε and each neighbourhood U_g of the point g.

- By $L_q^+(f, x_0)$ we denote the set of all right qualitative limit numbers of f at x_0.
- By $L_q^-(f, x_0)$ we denote the set of all left qualitative limit numbers of f at x_0.
- By $L_q(f, x_0)$ we denote the set of all qualitative limit numbers of f at x_0.

Then, similarly as for usual limit numbers one can state:

Theorem 5. *For arbitrary real function f on the interval (a,b) and any x_0 from (a,b) the sets $L_q(f, x_0)$, $L_q^-(f, x_0)$ and $L_q^+(f, x_0)$ are non-empty, closed and*

$$L_q(f, x_0) = L_q^-(f, x_0) \cup L_q^+(f, x_0).$$

Considering the sets of qualitative limit numbers we can get the analogue of Rome's Theorem, namely:

Theorem 6. *For any function $f\colon (a,b) \longrightarrow \mathbb{R}$ the set*

$$\{x \in (a,b) \,:\, L_q^-(f, x) \neq L_q^+(f, x)\}$$

is at most countable.

We can observe that the considered sets are at most countable, it means that they are rather small with natural topology in the set of real numbers. The quantity of such sets will be of our main interest. Unfortunately not always such sets must be countable.

2.3. **Approximate Limit Numbers.** Several mathematicians considered approximate limit numbers but we remind basic definitions and properties.

Definition 4. *A number g or $+\infty$ or $-\infty$ is called the approximate limit number of a function f at a point x_0 if the set*

$$\{x \in (a,b) \,:\, f^{-1}(U(g)) \cap [(x_0 - \varepsilon, x_0 + \varepsilon)]\}$$

has positive upper exterior density at x_0 *for every open neighbourhood* U_g *of the point* g *and each positive* ε*.*

Definition 5. *A number* g *or* $+\infty$ *or* $-\infty$ *is called the left approximate limit number of a function* f *at a point* x_0 *if the set*

$$\{x \in (a,b) \,:\, f^{-1}(U(g)) \cap [(x_0 - \varepsilon, x_0)]\}$$

has positive upper exterior density at x_0 *for every open neighbourhood* U_g *of the point* g *and each positive* ε*.*

And similarly, a number g (or $+\infty$, $-\infty$) is called the right approximate limit number of a function f at a point x_0 if the set

$$\{x \in (a,b) \,:\, f^{-1}(U(g)) \cap [(x_0, x_0 + \varepsilon)]\}$$

has positive upper exterior density at x_0 for every open neighbourhood U_g of the point g and each positive ε.

- By $L_a^+(f, x_0)$ we denote the set of all right approximate limit numbers of f at x_0.
- By $L_a^-(f, x_0)$ we denote the set of all left approximate limit numbers of f at x_0.
- By $L_a(f, x_0)$ we denote the set of all approximate limit numbers of f at x_0.

Then, similarly as for usual limit numbers one can state:

Theorem 7. *For arbitrary real function* f *on the interval* (a,b) *and any* x_0 *from* (a,b) *the sets* $L_a(f,x_0)$, $L_a^-(f,x_0)$ *and* $L_a^+(f,x_0)$ *are non-empty, closed and*

$$L_a(f,x_0) = L_a^-(f,x_0) \cup L_a^+(f,x_0).$$

Now considering the sets of approximate limit numbers we can get the analogue of Rome's Theorem, but:

Theorem 8. (M. Kulbacka [14]). *For any function* $f\colon (a,b) \longrightarrow \mathbb{R}$ *the set*

$$\{x \in (a,b) \,:\, L_a^-(f,x) \neq L_a^+(f,x)\}$$

is first category set and has measure 0*.*

This time the sets of the first category which have measure 0 do not characterize the set of asymmetry of functions. J. Jaskuła gave some additional properties for the set of approximate asymmetry.

Theorem 9. (J. Jaskuła [5]) *For any function* $f\colon (a,b) \longrightarrow \mathbb{R}$ *the set*

$$\{x \in (a,b) \,:\, L_a^-(f,x) \neq L_a^+(f,x)\}$$

is first category and has measure 0*, moreover it is of type* $F_{\sigma\delta\sigma}$*.*[1]

[1] W.Wilczyński informed me that the results of J. Jaskuła were a big deeper, i.e. the set approximate asymmetry is also σ-porous.

2.4. **Generalized Limit Numbers.** Let us observe that the class of sets which are of the first category at the point x_0 and the class of positive upper external density at that point have common properties. When we denote such a class by $\mathcal{B}$ then this class fulfils:

(1) If $B \in \mathcal{B}$ and $E \supset B$, then $E \in \mathcal{B}$,
(2) If $B_1 \cup B_2 \in \mathcal{B}$ then $B_1 \in \mathcal{B}$ or $B_2 \in \mathcal{B}$,
(3) If $B \in \mathcal{B}$ and $\varepsilon > 0$ then $B \cap (x_0 - \varepsilon, x_0 + \varepsilon) \in \mathcal{B}$.

The class of sets which are uncountable in each $(x_0 - \varepsilon, x_0 + \varepsilon)$ or have positive outer measure in each such interval and many other classes of sets have the previously pointed properties. The articles on this topic are as follows: J. Jędrzejewski [7], [8], J. Jędrzejewski with W. Wilczyński [13], J. Jędrzejewski with S. Kowalczyk [10] and [11].

Let us start now from the beginning:

Definition 6. *For each $x \in \mathbb{R}$ let $\mathfrak{B}_x^+$ be a class of non-empty sets fulfilling the following conditions:*

(1) $B_1 \cup B_2 \in \mathfrak{B}_x^+ \iff (B_1 \in \mathfrak{B}_x^+ \vee B_2 \in \mathfrak{B}_x^+)$,
(2) $B \cap (x, x+t) \in \mathfrak{B}_x^+$ *for each* $B \in \mathfrak{B}_x^+$ *and* $t > 0$.

For each $x \in \mathbb{R}$ let $\mathfrak{B}_x^-$ be a class of non-empty sets fulfilling the following conditions:

(1) $B_1 \cup B_2 \in \mathfrak{B}_x^- \iff (B_1 \in \mathfrak{B}_x^- \vee B_2 \in \mathfrak{B}_x^-)$,
(2) $B \cap (x, x+t) \in \mathfrak{B}_x^-$ *for each* $B \in \mathfrak{B}_x^-$ *and* $t > 0$.

Let $\mathfrak{B}_x = \mathfrak{B}_x^- \cup \mathfrak{B}_x^+$.

Definition 7. *If f defined in some (a,b) is a real function, then a number (or $+\infty$ or $-\infty$) is called $\mathfrak{B}$-limit number of f at x_0 from (a,b) if*

$$\left\{x \in (a,b) \,:\, f^{-1}(U_g)\right\} \in \mathfrak{B}_{x_0}$$

for any neighbourhood U_g of the point g.

Definition 8. *If*

$$\left\{x \in (a,b) \,:\, f^{-1}(U_g) \in \mathfrak{B}_{x_0}^-\right\}$$

for any neighbourhood U_g of the point g, then g is called the left $\mathfrak{B}$-limit number of a function f at a point x_0.

Similarly we define right $\mathfrak{B}$-limit numbers of a function f at a point x_0.

- By $L_{\mathfrak{B}}^+(f, x_0)$ we denote the set of all right $\mathfrak{B}$-limit numbers of f at x_0.
- By $L_{\mathfrak{B}}^-(f, x_0)$ we denote the set of all left $\mathfrak{B}$-limit numbers of f at x_0.
- By $L_{\mathfrak{B}}(f, x_0)$ we denote the set of all $\mathfrak{B}$-limit numbers of f at x_0.

Then, as for usual limit numbers, one can state:

Theorem 10. *For arbitrary real function f on the interval (a,b) and any x_0 from (a,b) the sets $L_{\mathfrak{B}}(f,x_0)$, $L^-_{\mathfrak{B}}(f,x_0)$ and $L^+_{\mathfrak{B}}(f,x_0)$ are non-empty, closed and*

$$L_{\mathfrak{B}}(f,x_0) = L^-_{\mathfrak{B}}(f,x_0) \cup L^+_{\mathfrak{B}}(f,x_0).$$

Considering the sets of $\mathfrak{B}$-limit numbers we are not able to get the analogue of Rome's Theorem. The situation depends on the class $\mathfrak{B}$. But if we add a special condition for the family $\mathfrak{B}$, we can get adequate analogue of Young's theorem.

Definition 9. *We say that the class $\mathfrak{B}$ fulfils condition $\mathscr{M}$ if*

$$\bigcup_{n=1}^{\infty} E_n \in \mathfrak{B}_{x_0}$$

for any: $x_0 \in (a,b)$, sequence $(x_n)_{n=1}^{\infty}$ converging to x_0 and every sequence of sets $(E_n)_{n=1}^{\infty}$ such that $E_n \in \mathfrak{B}_{x_n}$.

This condition permits us to state:

Theorem 11. *If the class $\mathfrak{B}$ fulfils condition $\mathscr{M}$, then*

$$\left\{x \in (a,b) \,:\, L^-_{\mathfrak{B}}(f,x) \neq L^+_{\mathfrak{B}}(f,x)\right\}$$

is at most countable set for any function $f\colon (a,b) \longrightarrow \mathbb{R}$.

3. Asymmetry for Some Classes of Functions

3.1. **Differentiation of Functions.** Everybody knows:

Theorem 12. *The set of all those points at which left derivative of a function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ is different from the right derivative of this function is at most countable.*

3.2. **Continuity of Functions.** One can get that the set of points at which a function is continuous from exactly one side as a quite simple corollary of Young's Theorem.

Theorem 13. *For any function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ the set of all points at which f is continuous from the only one side is at most countable.*

3.3. **Darboux Condition of Functions.** As before: everybody knows that Darboux condition has been originally defined as a global condition of a function. It sounds like this: the function f fulfils Darboux condition if it takes all values in between; exactly:

Definition 10. *We say that a function $f\colon (a,b) \longrightarrow \mathbb{R}$ fulfils Darboux condition if for any x_1 and x_2 such that $f(x_1) \neq f(x_2)$ and any number c lying between $f(x_1)$ and $f(x_2)$ there exists a point x lying (strictly) between x_1 and x_2 such that $f(x) = c$.*

This condition can be replaced by the one that function f transforms connected sets onto connected sets.

But still this condition is not good enough to say about asymmetry. We should define this condition locally, even more it must be defined separately for both sides. Let's start to do it, what was done by A. Bruckner and J. Ceder in 1965. For simplicity, let us assume that all the discussed functions are bounded.

Definition 11. (A. Bruckner, J. Ceder) [2]) *A function $f\colon (a,b) \longrightarrow \mathbb{R}$ is said to be Darboux from the left side at a point $x_0 \in (a,b)$ if*

(1) $f(x_0) \in L^-(f,x_0)$,
(2) *for each $c \in (\inf L^-(f,x_0), \sup L^-(f,x_0))$ and for each $t > 0$ there exists a point $x \in (x_0 - t, x_0)$ such that $f(x) = c$.*

Similarly,

Definition 12. *We say that a function $f\colon (a,b) \longrightarrow \mathbb{R}$ is Darboux from the right side at a point $x_0 \in (a,b)$ if*

(1) $f(x_0) \in L^+(f,x_0)$,
(2) *for each $c \in (\inf L^+(f,x_0), \sup L^+(f,x_0))$ and for each $t > 0$ there is a point $x \in (x_0, x_0 + t)$ such that $f(x) = c$.*

In the end:

Definition 13. *We say that a function $f\colon (a,b) \longrightarrow \mathbb{R}$ is Darboux at a point $x_0 \in (a,b)$ if it is Darboux from both sides at x_0.*

These definitions would not be good enough if the next theorem is false. But luckily it is not so.

Theorem 14. *A function $f\colon (a,b) \longrightarrow \mathbb{R}$ is Darboux if and only if it is Darboux at each point $x_0 \in (a,b)$.*

And now we can say about Darboux asymmetry.

Theorem 15. [9]. *For each function $f\colon (a,b) \longrightarrow \mathbb{R}$ the set of all those points at which f Darboux from exactly one side is at most countable.*

3.4. **Connectedness of Functions.** Next class of functions we want to discuss is the class of functions with connected graphs. They are called connected functions, however they can be defined in each topological spaces we shall consider only real functions defined in an interval. The adequate characterization has been given by B. D. Garret, D. Nelms and K. R. Kellum [3].

Definition 14. *A function $f\colon (a,b) \longrightarrow \mathbb{R}$ is called connected if its graph is a connected set on the plane.*

As before this definition is a global one, we have to find a local definition which will be as good as to get that local and global characterizations coincide.

As before, we assume that all discussed functions are bounded.

Definition 15. (B. D. Garret, D. Nelms, K. R. Kellum) [3]) *A function $f\colon (a,b) \longrightarrow \mathbb{R}$ is connected from the left side at a point $x_0 \in (a,b)$ if*

(1) $f(x_0) \in L^-(f, x_0)$,
(2) *for each continuum K (connected and compact set) such that*

$$\mathrm{proj}_x(K) = [x_0 - t, x_0] \quad \text{for some} \quad t > 0$$

and

$$\mathrm{proj}_y(K) \subset \big(\inf L^-(f, x_0), \sup L^-(f, x_0)\big)$$

the (graph) function f has common point with K.

Similarly:

Definition 16. *A function $f\colon (a,b) \longrightarrow \mathbb{R}$ is connected from the right side at a point $x_0 \in (a,b)$ if*

(1) $f(x_0) \in L^+(f, x_0)$,
(2) *for each continuum K such that*

$$\mathrm{proj}_x(K) = [x_0, x_0 + t] \quad \text{for some} \quad t > 0$$

and

$$\mathrm{proj}_y(K) \subset \big(\inf L^+(f, x_0), \sup L^+(f, x_0)\big)$$

the (graph) function f has common point with K.

Definition 17. *We say that a function $f\colon (a,b) \longrightarrow \mathbb{R}$ is connected at a point $x_0 \in (a,b)$ if it is connected from both sides at x_0.*

And of course:

Theorem 16. *A function $f\colon (a,b) \longrightarrow \mathbb{R}$ is connected if and only if it is connected at each point $x_0 \in (a,b)$.*

Finally, we are able to formulate theorem on connectivity asymmetry.

Theorem 17. *For each function $f\colon (a,b) \longrightarrow \mathbb{R}$ the set of all those points at which f is connected from exactly one side is at most countable.*

3.5. **Almost Continuity of Functions.** The last class of functions we want to discuss is the class of almost continuous functions. The adequate local characterization has been given by J. M. Jastrzębski, T. Natkaniec and J. Jędrzejewski [6].

Definition 18. *A function $f\colon (a,b) \longrightarrow \mathbb{R}$ is called almost continuous if each neighbourhood of its graph contains some continuous function defined in (a,b).*

As before this definition is a global one, we have to find a local definition which will be as good as to get that local and global characterizations coincide.

We assume that all discussed functions are bounded.

Definition 19. *A function $f\colon (a,b) \longrightarrow \mathbb{R}$ is almost continuous from the left side at a point $x_0 \in (a,b)$ if*

(1) $f(x_0) \in L^-(f,x_0)$,
(2) *there is a positive ε such that for each open neighbourhood of $f_{|(x,\infty)}$ arbitrary $y \in (\inf L^-(f,x_0), \sup L^-(f,x_0))$, arbitrary neighbourhood G of the point $(x,y) \in \mathbb{R}^2$ and arbitrary $t \in (x_0, x_0+\varepsilon)$ there is a continuous function $g\colon (x_0, x_0+\varepsilon) \longrightarrow \mathbb{R}$ such that $g \subset U \cup G$ and $g(x_0) = y$, $g(t) = f(t)$.*

Similarly one can define almost continuity from the right side at a point $x_0 \in (a,b)$.

Definition 20. *We say that a function $f\colon (a,b) \longrightarrow \mathbb{R}$ is almost continuous at a point $x_0 \in (a,b)$ if it is almost continuous from both sides at x_0.*

And of course:

Theorem 18. *A function $f\colon (a,b) \longrightarrow \mathbb{R}$ is almost continuous if and only if it is almost continuous at each point $x_0 \in (a,b)$.*

Finally, one can state:

Theorem 19. *For each function $f\colon (a,b) \longrightarrow \mathbb{R}$ the set of all those points at which f is almost continuous from exactly one side is at most countable.*

4. General Approach to Asymmetry of Functions

Some general theorems were discussed in previous parts of the article. Let us come to Thomson's monograph. B. S. Thomson gathered several ideas in one theory. He defined local systems which contain $\mathfrak{B}$ classes and $\mathfrak{B}^*$ classes that have been defined in [7]. For sake of completeness let us remind the basic notions.

4.1. **Local Systems.**

Definition 21. B. S. Thomson [17].
By a local system in $\mathbb{R}$ we mean a class $\mathcal{S}$ consisting of non-empty collections $\mathcal{S}(x)$ for each real number x, fulfilling the following conditions:

$$\{x\} \notin \mathcal{S}(x), \tag{1}$$

$$E \in \mathcal{S}(x) \Longrightarrow x \in E, \tag{2}$$

$$(E \in \mathcal{S}(x) \wedge F \supset E) \Longrightarrow F \in \mathcal{S}(x), \tag{3}$$

$$(E \in \mathcal{S}(x) \wedge \delta > 0) \Longrightarrow E \cap (x-\delta, x+\delta) \in \mathcal{S}(x). \tag{4}$$

Definition 22. *By a left local system in $\mathbb{R}$ we mean a class $\mathcal{S}$ consisting of non-empty collections $\mathcal{S}(x)$ for each real number x, fulfilling the following conditions:*

$$\{x\} \notin \mathcal{S}(x), \tag{5}$$

$$E \in \mathcal{S}(x) \Longrightarrow x \in E, \tag{6}$$

$$(E \in \mathcal{S}(x) \wedge F \supset E) \Longrightarrow F \in \mathcal{S}(x), \tag{7}$$

$$(E \in \mathcal{S}(x) \wedge \delta > 0) \Longrightarrow E \cap (x-\delta, x] \in \mathcal{S}(x). \tag{8}$$

Similarly we define right local systems.

A local system is called filtering at a point x if

$$E \cap F \in \mathcal{S}(x) \ \text{ whenever } \ E \in \mathcal{S}(x) \ \text{ and } \ F \in \mathcal{S}(x). \tag{9}$$

A local system is called filtering if it is filtering at each x in $\mathbb{R}$.
A local system is called bilateral if

$$E \cap (x-\delta, x) \neq \varnothing \quad \text{and} \quad E \cap (x, x+\delta) \neq \varnothing$$

for each $x \in \mathbb{R}$, $E \in \mathcal{S}(x)$ and $\delta > 0$.

Let us observe that those definitions are very close to Definition 6. When B. S. Thomson assumes that dual system for $\mathcal{S}$ is filtering, then $\mathcal{S}$ fulfils all requirements of Definition 6. The only difference lays in the belonging of the point x to every set from the class $\mathcal{S}_x$.

Definition 23. *A number g is called $\mathcal{S}$-limit of a function f at a point x if*

$$f^{-1}(g-\varepsilon, g+\varepsilon) \cup \{x\} \in \mathcal{S}(x)$$

for each positive ε.

We shall write then

$$g = (\mathcal{S}) \lim_{t \to x} f(t).$$

The set of all $(\mathcal{S})$-limits are denoted by $\Lambda_{\mathcal{S}}(f, x)$.

For each local system $\mathcal{S}$ there is a system $\mathcal{S}^*$ which is also a local system, that is defined by:

$$E \in \mathcal{S}^*(x) \iff \big(x \in E \wedge \big[(\mathbb{R} \setminus E) \cup \{x\}\big] \notin \mathcal{S}(x)\big).$$

This system is called dual system for $\mathcal{S}$.

A system $\mathcal{S}$ is called filtering if $E_1 \cap E_2 \in \mathcal{S}(x)$ for every sets $E_1 \in \mathcal{S}(x)$ and $E_2 \in \mathcal{S}(x)$ and each $x \in \mathbb{R}$.

Definition 24. *We say that two systems $\mathcal{S}_1$ and $\mathcal{S}_2$ satisfy a joint intersection condition if for any choices $\{E_x : x \in \mathbb{R}\}$ and $\{D_x : x \in \mathbb{R}\}$ such that $E_x \in \mathcal{S}_1(x)$, $D_x \in \mathcal{S}_2(x)$ there exists a gauge δ on $\mathbb{R}$ so that if $0 < |x-y| < \min\{\delta(x), \delta(y)\}$ then at least one of the sets $E_x \cap D_y$ or $D_x \cap E_y$ contains points other than x and y.*

By a gauge on the set $\mathbb{R}$ we mean a positive function defined in $\mathbb{R}$.

And now we are able to formulate the asymmetry theorem given by Thomson.

Theorem 20. *Let $\mathcal{S}^1$, $\mathcal{S}^2$ be local systems such that both of them are filtering and that the pair $(\mathcal{S}_1, \mathcal{S}_2)$ has the joint intersection condition. Then for any function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ the set*

$$\{x \in \mathbb{R} \,:\, \Lambda_{\mathcal{S}^1}(f,x) \neq \Lambda_{\mathcal{S}^2}(f,x)\}$$

is at most countable.

Example 2.

Let $\mathcal{S}^1_x$ be the class consisting of all sets E for which $E \cap (x-\varepsilon, x+\varepsilon)$ is of the first category.

Let $\mathcal{S}^2_x$ be the class consisting of all sets D for which $D \cap (x-\varepsilon, x+\varepsilon)$ has positive outer measure.

There are two sets A and B such that $A \cap B = \varnothing$, $A \cup B = (0,1)$, A is of the first category in $(0,1)$, and B has measure 1.

Let $f\colon (0,1) \longrightarrow \mathbb{R}$ be defined as follows:

$$f(x) = \begin{cases} 0 & \text{if} \quad x \in A, \\ 1 & \text{if} \quad x \in B. \end{cases}$$

For this function, all points from $(0,1)$ are points of $\left(\mathcal{S}^1, \mathcal{S}^2\right)$-asymmetry.

4.2. **Świątkowski Approach to Asymmetry.**

Definition 25. (T. Świątkowski [15]) *Let T be a stronger topology in $\mathbb{R}$ than the natural one. For a subset E of $\mathbb{R}$ the symbol E'_T denote the set of all accumulation points with respect to topology T. Let moreover $L_x = (-\infty, x)$ and $P_x = (x, \infty)$ for any real number x. Consider now the function φ in the following way:*

$$x \in \varphi(A) \quad if \quad x \in (A \cap L_x)'_T \triangle (A \cap P_x)'_T$$

for any subset A of $\mathbb{R}$.

Each point from the set $(A \cap L_x)'_T \triangle (A \cap P_x)'_T$ is called T-asymmetry point of the set A.

Definition 26. *Let $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ be arbitrary function and x a real number. We say that g is T-limit number of the function f at a point x if*

$$x \in \left(f^{-1}(U)\right)'_T$$

for each neighbourhood U of the point x.

Not every topology is good enough to get the adequate theorem on asymmetry; let us call the property (W) from the article [15].

Definition 27. [15] *Let T be a stronger topology then the natural one in the set $\mathbb{R}$. We say that T fulfils condition (W) if for every $x \in \mathbb{R}$, sequence $(x_n)_{n=1}^{\infty}$ converging to x and every sequence $(E_n)_{n=1}^{\infty}$ such that $x_n \in (E_n)'_T$ the point x belongs to $(\bigcup_{n=1}^{\infty} E_n)'_T$.*

This condition (W) for the topology T described as above is equivalent to the condition (W'):
for an arbitrary $x \in \mathbb{R}$ and its T-neighbourhood U there exists a positive number δ such that $((x-\delta, x+\delta) \setminus U)'_T = \varnothing$.

The condition (W') allows to formulate one of the most general theorems on asymmetry.

Theorem 21. *If T is a stronger than the natural topology in the set $\mathbb{R}$ and fulfils condition (W), then for any function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ the set of asymmetry of f is at most countable.*

It is now easy to observe that:

If T is a natural topology in $\mathbb{R}$, Theorem 21 allows us to obtain the classical Young's Theorem on asymmetry. It is implied from the fact that T fulfils condition (W') (see Theorem 3).

Let us remark that if T is a Hashimoto topology in $\mathbb{R}$ generated by sets of the first category, Theorem 21 allows us to obtain Theorem on qualitative asymmetry of functions. It follows from the fact that T also fulfils (W') (see Theorem 6).

4.3. **Comments on the Three Approaches to Asymmetry.** When we want to compare the three ideas of B. S. Thomson, of T. Świątkowski and J. Jędrzejewski, we can observe that some local systems $\mathcal{S}$-limits can be understood as $\mathfrak{B}$-limits, some systems can be understood as systems $\mathfrak{B}$. However, in each theorem where Thomson assumes that the dual system for a system $\mathcal{S}$ is filtering, then the system fulfils all conditions for the system $\mathfrak{B}$. Świątkowski's condition and mine called $\mathscr{W}$ or $\mathscr{M}$ are equivalent, so Thomson's theorems are almost the same as Świątkowski's and mine ones. The only difference lays on different approaches to the problem.

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SOME REMARKS ON STRONG SEQUENCES

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Abstract

Strong sequences were introduced by Efimov in the 60s' of the last century as a useful method for proving well known theorems on dyadic spaces i.e. continuous images of the Cantor cube. The aim of this paper is to show relations between the cardinal invariant associated with strong sequences and well known invariants of the continuum.

1. Introduction

Strong sequences were introduced by B. A. Efimov in [4], as a useful tool for proving well known theorems on dyadic spaces. Among others he proved that strong sequences do not exist in the subbase of the Cantor cube. This is our opinion that it could be interesting the answer of the natural question about properties of spaces in which strong sequences exist and consequences of such existence. This is how the interest of the strong sequences method was born, (for further historical notes concerning strong sequences see [6]). Particularly, strong sequences method, as was shown in e.g. [7, 8] is equivalent to partition theorems. Moreover, if we associate the cardinal invariant with the length of strong sequences in spaces where such sequences exist, we can obtain interesting results, (see also [8, 9]). This is our hope that this invariant can be usefull characterisation of such spaces.

In this paper we will consider the space $(\omega^\omega, \leq^*)$ in which, as we will show, strong sequences exist. We will investigate inequalities between invariant $\hat{\mathfrak{s}}$ associated with strong sequences and other well known invariants like: boundeness, covering number and the invariant associated with MAD families.

Our paper is organized as follows. In section 2 we gather all definitions and previous facts needed for further parts of this paper. In Section 3 we show main results. The paper is finished by some results in forcing,

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(Section 4) in which we will show some strong inequalities which can be obtained between $\hat{\mathfrak{s}}$ and considered invariants. In this part we give some open problems.

2. Definitions and previous results

1. Consider a *partially preordered* set $(X, \preceq)$, i.e. a set ordered by reflexive and transitive relation $\preceq$. Let $a, b, c, x \in X$. We say that a and b are *comparable* iff $a \preceq b$ or $b \preceq a$. We say that a and b are *compatible* iff there exists $c \in X$ such that $a \preceq c$ and $b \preceq c$. (In this case we say that a, b have a *bound*). A set $A \subset X$ is called an *ω-directed set* iff every subset of A of cardinality less than ω has a bound which belongs to A.

Definition 1. *A sequence $(S_\phi, H_\phi)_{\phi<\alpha}$, where $S_\phi, H_\phi \subset X$, and $|S_\phi| < \omega$ is called a strong sequence if:*
1^o *$S_\phi \cup H_\phi$ is ω-directed for all $\phi < \alpha$;*
2^o *$S_\psi \cup H_\phi$ is not ω-directed, for all ψ and ϕ such that $\phi < \psi < \alpha$.*

In [6] the strong sequence number $\hat{\mathfrak{s}}(X)$ was introduced as follows:

$$\hat{\mathfrak{s}}(X) = \sup\{\kappa\colon \text{there exists a strong sequence on } X \text{ of length } \kappa\}. \tag{1}$$

2. We say that $(X, \preceq)$ iff $\preceq$ is reflexive and transitive.
A subset $B \subset X$ is called *bounded* iff B has a bound. The set which is not bounded will be called *unbounded.*
A subset $A \subseteq B \subseteq X$ is called *cofinal* in B iff for any $b \in B$ there exists $a \in A$ such that $b \preceq a$. A cofinal subset in the whole set X is called also *a dominating set.* The following invariants are well known:

$$\mathfrak{b}(X) = \min\{|A|\colon A \subset X \wedge A \text{ is unbounded in } X\}, \tag{2}$$

$$\mathfrak{d}(X) = \min\{|A|\colon A \subset X \wedge A \text{ is cofinal in } X\}. \tag{3}$$

Fact 1 ([3]). *Let $(X, \preceq)$ be a partially preordered set without maximal elements. Then $\mathfrak{b}(X)$ is regular and*

$$\mathfrak{b}(X) \leq cf(\mathfrak{d}(X)) \leq \mathfrak{d}(X). \tag{4}$$

3. We will provide our considerations for $(X, \preceq) = (\omega^\omega, \leq^*)$, i.e. in the set of all functions $\omega \to \omega$ ordered by

$$f \leq^* g \text{ iff } |\{n \in \omega\colon g(n) < f(n)\}| < \omega. \tag{5}$$

We accept the notation: $\hat{\mathfrak{s}} = \hat{\mathfrak{s}}(\omega^\omega), \mathfrak{b} = \mathfrak{b}(\omega^\omega), \mathfrak{d} = \mathfrak{d}(\omega^\omega)$.

4. A family $\mathcal{I}$ of subsets of X which satisfies the following three conditions

1) $A \in \mathcal{I}$ and $B \subset A$ then $B \in \mathcal{I}$;
2) $\{x\} \in \mathcal{I}$ for all $x \in \mathcal{I}$;
3) $X \notin \mathcal{I}$
is called *a family of thin sets.*
A subfamily $\mathcal{B} \subset \mathcal{I}$ is called *a base* of the family $\mathcal{I}$ of thin sets iff for each set $A \in \mathcal{I}$ there exists a set $B \in \mathcal{B}$ such that $A \subseteq B$.

We remind definitions of the following invariants, (see e. g. [3] p.250):

$$\operatorname{add}(\mathcal{I}) = \min\left\{|\mathcal{A}| \colon \mathcal{A} \subset \mathcal{I} \wedge \bigcup \mathcal{A} \notin \mathcal{I}\right\} \tag{6}$$

$$\operatorname{cov}(\mathcal{I}) = \min\left\{|\mathcal{A}| \colon \mathcal{A} \subset \mathcal{I} \wedge \bigcup \mathcal{A} = X\right\} \tag{7}$$

$$\operatorname{non}(\mathcal{I}) = \min\left\{|A| \colon A \notin \mathcal{I} \wedge A \in \mathcal{P}(X)\right\} \tag{8}$$

$$\operatorname{cof}(\mathcal{I}) = \min\left\{|\mathcal{A}| \colon \mathcal{A} \subset \mathcal{I} \wedge \mathcal{A} \text{ is a base of } \mathcal{I}\right\}. \tag{9}$$

Notice that any ideal on X is a family of thin sets. (Clearly, $\mathcal{I}$ is an ideal iff $\operatorname{add}(\mathcal{I}) \geq \aleph_0$).

The following diagram is known in the literature as "Cichoń diagram" and was introduced by Fremlin in [5]. Since that paper the diagram has been completed and modified by many authors. Below we remind this diagram for four invariants defined above.

Fact 2 ([1])**.** *If $\mathcal{I}$ is a family of thin sets, then*

$$\begin{tikzcd}
 & & \operatorname{cov}(\mathcal{I}) \arrow[dr] & & \\
\aleph_0 \arrow[r] & \operatorname{add}(\mathcal{I}) \arrow[ur] \arrow[dr] & & \operatorname{cof}(\mathcal{I}) \arrow[r] & 2^{\aleph_0} \\
 & & \operatorname{non}(\mathcal{I}) \arrow[ur] & &
\end{tikzcd}$$

where $\alpha \to \beta$ denotes $\alpha \leq \beta$.

5. Let $\mathbb{R}$ be the real line with standard topology. Let μ be the Lebesque measure on $\mathbb{R}$. Then

$$\mathcal{M} = \{A \subset \mathbb{R} \colon A \text{ is meager}\}, \tag{10}$$

$$\mathcal{N} = \{A \subset \mathbb{R} \colon \mu(A) = \emptyset\}. \tag{11}$$

Notice, that $\mathcal{M}$ and $\mathcal{N}$ are both ideals.

6. In [1] one can find the following results:

Fact 3 (Bartoszyński) *$cov(\mathcal{M})$ is the cardinality of the smallest family $\mathcal{F} \subseteq \omega^\omega$ such that*

$$\forall_{g\in\omega^\omega}\exists_{f\in\mathcal{F}}|\{n \in \omega\colon f(n) \neq g(n)\}| < \omega. \tag{12}$$

Fact 4 (Keremedis) *$non(\mathcal{M})$ is the cardinality of the smallest family $\mathcal{F} \subseteq \omega^\omega$ such that*

$$\forall_{g\in\omega^\omega}\exists_{f\in\mathcal{F}}|\{n \in \omega\colon f(n) = g(n)\}| < \omega. \tag{13}$$

Fact 5 (Rothberger)

$$\mathrm{cov}(\mathcal{M}) \leq \mathrm{non}(\mathcal{N}) \text{ and } \mathrm{cov}(\mathcal{N}) \leq \mathrm{non}(\mathcal{M}). \tag{14}$$

Fact 6 (Bartoszyński, Raisonnier and Stern)

$$\mathrm{add}(\mathcal{N}) \leq \mathrm{add}(\mathcal{M}), \tag{15}$$

$$\mathrm{cof}(\mathcal{M}) \leq \mathrm{cof}(\mathcal{N}). \tag{16}$$

Fact 7 (Miller, Truss)

$$\mathrm{add}(\mathcal{M}) = \min\{\mathrm{cov}(\mathcal{M}), \mathfrak{b}\}. \tag{17}$$

Fact 8 (Fremlin)

$$\mathrm{cof}(\mathcal{M}) = \max\{\mathrm{non}(\mathcal{M}), \mathfrak{d}\}. \tag{18}$$

According to equalities (14) - (18) the following diagram holds:

Fact 9 ([1]).

$$\begin{tikzcd}
\mathrm{cov}(\mathcal{N}) \arrow[r] & \mathrm{non}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{N}) \\
 & \mathfrak{b} \arrow[u] \arrow[r] & \mathfrak{d} \arrow[u] & \\
\mathrm{add}(\mathcal{N}) \arrow[r] \arrow[uu] & \mathrm{add}(\mathcal{M}) \arrow[r] \arrow[u] & \mathrm{cov}(\mathcal{M}) \arrow[r] \arrow[u] & \mathrm{non}(\mathcal{N}) \arrow[uu]
\end{tikzcd}$$

where $\alpha \to \beta$ denotes $\alpha \leq \beta$.

Observation 1. ***(i)*** *Let $\mathcal{F} \subseteq \omega^\omega$ be the smallest family of the property*

$$\forall_{g\in\omega^\omega}\exists_{f\in\mathcal{F}} |\{n \in \omega\colon f(n) = g(n)\}| < \omega.$$

Then $|\{n \in \omega\colon f_\alpha(n) \neq f_\beta(n)\}| = \omega$ for all $f_\alpha, f_\beta \in \mathcal{F}, \alpha \neq \beta$.
(ii) *Let $\mathcal{F} \subseteq \omega^\omega$ be the smallest family of the property*

$$\forall_{g\in\omega^\omega}\exists_{f\in\mathcal{F}} |\{n \in \omega\colon f(n) \neq g(n)\}| < \omega.$$

Then $|\{n \in \omega\colon f_\alpha(n) \neq f_\beta(n)\}| = \omega$ for all $f_\alpha, f_\beta \in \mathcal{F}, \alpha = \beta$.

Proof. We prove (i) only, (ii) can be proved similarly but using Fact 3.
(i) By Fact 4 we have $|\mathcal{F}| = \mathrm{non}(\mathcal{M})$. Suppose in contrary that there are $\alpha \neq \beta$ such that $|\{n \in \omega\colon f_\alpha(n) = f_\beta(n)\}| = \omega$. Let

$$A(\alpha, \beta) = \{n \in \omega\colon f_\alpha(n) = f_\beta(n)\}.$$

Let $\{g_\gamma \in \omega^\omega \setminus \mathcal{F}\colon \gamma < \eta\}$ be a family such that

$$|\{n \in \omega\colon g_\gamma(n) = f_\beta(n)\}| < \omega$$

for all $\gamma < \eta$. Let $B(\gamma, \beta) = \{n \in \omega\colon g_\gamma(n) = f_\beta(n)\}$ for all $\gamma < \eta$. Obviously $|A(\alpha, \beta) \cap B(\gamma, \beta)| < \omega$. Then $g_\gamma(n) = f_\alpha(n)$ for all $n \in A(\alpha, \beta) \cap B(\gamma, \beta)$. A contradiction with the minimality of $\mathcal{F}$. □

7. Two functions $f, g \in \omega^\omega$ are *almost disjoint* iff there are finite values of $\alpha \in \mathrm{Dom}(f) \cap \mathrm{Dom}(g)$ such that $f(\alpha) = g(\alpha)$. When the functions have domain ω almost disjointness means that they are eventually different ($f(\alpha) \neq g(\alpha)$) for all sufficiently large $\alpha < \omega$. A *maximal almost disjoint (MAD) family* of functions on ω is an almost disjoint family of functions $\omega \to \omega$ that is not properly included in another such family. In [2] the following invariant is associated with MAD families of functions:

(19) $$\mathfrak{a}_e = \min\{\mathcal{A} \subseteq P(\omega^\omega) : \mathcal{A} \text{ is a MAD family}\}.$$

Fact 10 ([2]).

(20) $$\mathfrak{a}_e \geq \omega^+.$$

Observation 2.

(21) $$\mathrm{non}(\mathcal{M}) \leq \mathfrak{a}_e.$$

Proof. Immediately by Fact 4. □

3. Main results

Theorem 1.

$$\mathfrak{b} \leq \hat{s}. \tag{22}$$

Proof. Suppose that $\hat{\mathrm{s}} < \mathfrak{b}$ and $\kappa \leq \hat{\mathrm{s}}$. Let $\{(S_\alpha, H_\alpha) : \alpha < \kappa\}$ be a maximal strong sequence in ω^ω. For any $\alpha < \kappa$ define

$$A_\alpha = \{f \in S_\alpha \setminus H_\beta \colon \{f\} \cup H_\beta \text{ is not } \omega\text{-directed for } \beta < \alpha\}.$$

Define an increasing function

$$F \colon \kappa \to \bigcup_{\alpha<\kappa} (S_\alpha \cup H_\alpha).$$

such that

$$F(\alpha) = \begin{cases} f_\alpha \in H_\alpha & \text{for } \alpha = 0; \\ f_\alpha \in A_\alpha & \text{for } \alpha > 0. \end{cases}$$

Since ω^ω has no maximal elements, this function is well-defined.

Let

$$A = \{f_\alpha \in A_\alpha \colon f_\alpha = F(\alpha), \alpha < \kappa\}.$$

Since $\kappa < \mathfrak{b}$, there exists $g \in A$ such that $f_\alpha \leq g$, for all $f_\alpha \in S_\alpha$. As ω^ω has no maximal elements, there exists $h \in \omega^\omega \setminus \bigcup_{\alpha<\kappa} (S_\alpha \cup H_\alpha)$ such that $g < h$. Thus there exists a maximal ω-directed set $S \subset \omega^\omega \setminus \bigcup_{\alpha<\kappa} (S_\alpha \cup H_\alpha)$ such that $h \in S$ and $S \cup H_\alpha$ is not ω-directed for any $\alpha < \kappa$. A contradiction with maximality of the strong sequence $\{(S_\alpha, H_\alpha) : \alpha < \kappa\}$. □

Theorem 2.

$$cov(\mathcal{M}) \leq \hat{s}. \tag{23}$$

Proof. Let $\operatorname{cov}(\mathcal{M}) = \kappa$. By Fact 3 there exists the smallest family

$$\mathcal{F} = \{f_\alpha \in \omega^\omega \colon \alpha < \kappa\}$$

fulfilling (12)

Thus we can construct a function $H \colon \omega^\omega \to \kappa$ such that

$$H(g) = \min\{\alpha \colon |\{n \in \omega \colon f_\alpha(n) = g(n)\}| = \omega\}.$$

The family $\mathcal{F}$ is well-ordered hence the function H is well-defined.
We will construct a strong sequence in ω^ω with relation defined as follows:

$$\text{if } f_\alpha \in \mathcal{F}, \text{ then } f_\alpha \preceq g \text{ iff } h(g) = \alpha;$$

$$\text{if } f \notin \mathcal{F}, \text{ then } f \preceq g \text{ iff } |\{n \in \omega \colon f(n) = g(n)\}| = \omega.$$

Let $g_0 \in \omega^\omega$ be an arbitrary function. Then there exists $f \in \mathcal{F}$ such that $|\{n \in \omega \colon f(n) = g_0(n)\}| = \omega$. Let $f_{\alpha_0} \in \mathcal{F}$ be a function such that $h(g_0) = \alpha_0$. Let $\mathcal{S}_0 = \{g_0\}$ and $\mathcal{H}_0 = \{g \in \omega^\omega \colon h(g) = \alpha_0\}$. Obviously $\mathcal{H}_0$ is non-empty. Let $(\mathcal{S}_0, \mathcal{H}_0)$ be the first element of a strong sequence.

Since $\mathcal{H}_0 \neq \omega^\omega$ there exists $g_1 \in \omega^\omega \setminus \mathcal{H}_0$ such that $h(g_1) \neq \alpha_0$. Hence we can construct the next element of the strong sequence. Let $f_{\alpha_1} \in \mathcal{F}$ be a fucntion such that $|\{n \in \omega\colon g_1(n) = f_{\alpha_1}(n)\}| = \omega$. Let $\mathcal{S}_1 = \{g_1\}$ and $\mathcal{H}_1 = \{g \in \omega^\omega \setminus \mathcal{H}_0\colon h(g) = \alpha_1\}$.

Assume that the strong sequence $\{(\mathcal{S}_\gamma, \mathcal{H}_\gamma) : \gamma < \beta\}$ such that

$$(\mathcal{S}_\gamma, \mathcal{H}_\gamma) = \left(\{g_\gamma\}, \left\{g \in \omega^\omega \setminus \bigcup \{\mathcal{H}_\delta\colon \delta < \gamma\} : h(g) = \alpha_\gamma\right\}\right),$$

where $g_\gamma \in \omega^\omega \setminus \bigcup_{\delta<\gamma} H_\delta$, has been defined,.

Since $\beta < \kappa$ and by Observation 1, there exists $g_\beta \in \omega^\omega \setminus \left\{f_{\alpha_\gamma}\colon \gamma < \beta\right\}$ be a function such that $|\left\{n \in \omega\colon g_\beta(n) = f_{\alpha_\beta}(n)\right\}| = \omega$. Let

$$(\mathcal{S}_\beta, \mathcal{H}_\beta) = \left(\{g_\beta\}, \left\{g \in \omega^\omega \setminus \bigcup\{\mathcal{H}_\gamma\colon \gamma < \beta\right\} : h(g) = \alpha_\beta\}\right).$$

Thus the strong sequence of length $|\mathcal{F}|$ has been constructed. □

Theorem 3.

$$\mathfrak{a}_e \leq \hat{s}. \tag{24}$$

Proof. By Fact 8 we have $\mathfrak{a}_e \geq \omega^+$. Let $\mathcal{F}_e$ be a MAD family of functions $\omega \to \omega$ of cardinality ω^+. We will construct a strong sequence of cardinality ω^+ in ω^ω with the following relatio:

$$f \preceq g \text{ iff } |\{\alpha \in \omega\colon f(\alpha) = g(\alpha)\}| = \omega.$$

Let $f_0 \in \mathcal{F}_e$ be a function. Let $(\mathcal{S}_0, \mathcal{H}_0) = (\{f_0\}, \{g \in \omega^\omega\colon f_0 \preceq g\})$ be the first element of a strong sequence. Obviously $(\mathcal{S}_0, \mathcal{H}_0)$ is non-empty because $f_0 \in \mathcal{H}_0$. Let $f_1 \in \mathcal{F}_e \setminus \mathcal{H}_0$. Let $(\mathcal{S}_1, \mathcal{H}_1) = (\{f_1\}, \{g \in \omega^\omega\colon f_1 \preceq g\})$. By our construction $\mathcal{H}_0 \cup \mathcal{H}_1$ is not ω-directed. Let $(\mathcal{S}_1, \mathcal{H}_1)$ be the second element of the strong sequence.

Assume that the strong sequence $\{(\mathcal{S}_\gamma, \mathcal{H}_\gamma) : \gamma < \beta < \omega^+\}$ such that

$$(\mathcal{S}_\gamma, \mathcal{H}_\gamma) = \left(\{f_\gamma\}, \left\{g \in \omega^\omega \setminus \bigcup\{\mathcal{H}_\delta\colon \delta < \gamma\colon f_\gamma \preceq g\right\}\right),$$

where $f_\gamma \in \mathcal{F}_e \setminus \bigcup\{\mathcal{H}_\delta\colon \delta < \gamma\}$, has been defined.

Since $\beta < \omega^+$ there exists $f_\beta \in \mathcal{F}_e \setminus \bigcup\{\mathcal{H}_\gamma\colon \gamma < \beta\}$. Let

$$(\mathcal{S}_\beta, \mathcal{H}_\beta) = \left(\{f_\beta\}, \left\{g \in \omega^\omega \setminus \bigcup\{\mathcal{H}_\delta\colon \gamma < \beta\colon f_\beta \preceq g\right\}\right),$$

Thus the strong sequence of length $|\mathcal{F}|$ has been constructed. □

Corollary 1.

$$non(\mathcal{M}) \leq \mathfrak{a}_e \leq \hat{s}. \tag{25}$$

Proof. Immediately by Fact 10 and Theorem 3. □

Theorem 4. *In $(\omega^\omega, \leq^*)$ there exists a strong sequence of length $2^{\aleph_0}$.*

Proof. Fix a MAD family of sets $\mathcal{A} = \left\{A_\alpha \subseteq [\omega]^\omega : \alpha < 2^{\aleph_0}\right\}$, (i.e. a family of infinite subsets of ω such that $|A \cap B| < \omega$ for any $A, B \in \mathcal{A}$). For each $A \in \mathcal{A}$ consider functions: $F_n^A \in \omega^\omega$ such that

$$F_n^A(a) = \begin{cases} n+1 & \text{for } a \in A \\ 0 & \text{for } a \notin A \end{cases}$$

and $F_\omega^A \in \omega^\omega$ such that

$$F_\omega^A(a) = \begin{cases} a & \text{for } a \in A \\ 0 & \text{for } a \notin A. \end{cases}$$

Obviously

$$F_0^A <^* F_1^A <^* \ldots <^* F_\omega^A.$$

Now take $(S_A, H_A) = \left(\{F_\omega^A\}, \left\{F_n^A \colon n < \omega\right\}\right)$. Then $S_A \cup H_A$ is ω-directed, because F_ω^A is its bound. Now take $A_\alpha, A_\beta \in \mathcal{A}$ such that $\alpha < \beta$. Then $S_{A_\beta} \cup H_{A_\alpha}$ is not ω-directed, because it contains no bound for H_{A_α}. Since all MAD families have cardinality $2^{\aleph_0}$ we obtain that $\{(S_A, H_A) : A \in \mathcal{A}\}$ is the required strong sequence. $\square$

Corollary 2. *The following diagram holds*

$$\begin{tikzcd}
 & non(\mathcal{M}) \arrow[rrr] \arrow[dr] & & & cof(\mathcal{M}) \arrow[r] & 2^{\aleph_0} \\
 & & \mathfrak{a}_e \arrow[r] & \hat{s} \arrow[urr] & & \\
 & \mathfrak{b} \arrow[uu] \arrow[rrr] & & & \mathfrak{d} \arrow[uu] & \\
\aleph_0 \arrow[r] & add(\mathcal{M}) \arrow[u] \arrow[rrr] & & & cov(\mathcal{M}) \arrow[u] \arrow[uul] &
\end{tikzcd}$$

where $\alpha \to \beta$ means $\alpha \leq \beta$:

Proof. Immediately by equalities (4), (17), (18) and Theorems 1-4. $\square$

4. Some results for forcing notion

According to [1] pp. 380-397, the following inequalities are consistent with ZFC.

In the iterated Cohen's model with finite supports $\mathrm{non}(\mathcal{M}) = \aleph_1 \wedge \mathrm{cov}(\mathcal{M}) = \mathfrak{c}$ which is connecting with Cichoń diagram we have $\mathrm{add}(\mathcal{N}) = \mathrm{add}(\mathcal{M}) = \mathrm{cov}(\mathcal{M}) = \mathrm{non}(\mathcal{M}) = \mathfrak{b} = \aleph_1$ and $\mathrm{cov}(\mathcal{M}) = \mathfrak{r} = \mathrm{cof}(\mathcal{M}) = \mathrm{cof}(\mathcal{N}) = \mathrm{non}(\mathcal{N}) = \mathfrak{c} > \aleph_1$. Thus

$$\mathrm{add}(\mathcal{N}) = \mathrm{add}(\mathcal{M}) = \mathrm{cov}(\mathcal{N}) = \mathrm{cov}(\mathcal{M}) = \mathfrak{b} < \hat{\mathrm{s}}. \tag{26}$$

By adding $\aleph_2$ random reals a model of CH we have $\mathrm{non}(\mathcal{N}) = \aleph_1 < \mathrm{cov}(\mathcal{N}) = \aleph_2 = \mathfrak{c}$. Thus

$$\mathrm{non}(\mathcal{N}) < \hat{\mathfrak{s}}. \tag{27}$$

By adding $\aleph_2$ Hechler's reals (with finite support) to a model of CH we get $\mathrm{cov}(\mathcal{N}) = \aleph_1 < \mathrm{add}(\mathcal{M}) = \aleph_2 = \mathfrak{c}$. Hence it is consistent that

$$\mathrm{cov}(\mathcal{N}) < \hat{\mathfrak{s}}. \tag{28}$$

Alternatively adding $\aleph_2$ Cohen and Laver reals (with countable support) over a model of CH we have $\mathrm{cov}(\mathcal{N}) = \aleph_1 < \mathrm{add}(\mathcal{M}) = \aleph_2 = \mathfrak{c}$ Thus

$$\mathrm{cov}(\mathcal{N}) < \hat{\mathfrak{s}}. \tag{29}$$

Alternatively iterating $\aleph_2$ times rational perfect forcing and Roslanowski-Shelah forcing over a model of CH we obtain $\aleph_1 = \mathrm{non}(\mathcal{M}) < \mathrm{non}(\mathcal{N}) = \mathfrak{d} = \aleph_2$. Therefore, it is consistent that

$$\mathrm{cov}(\mathcal{M}) < \hat{\mathfrak{s}}. \tag{30}$$

Finally in the iterated Sachs model we have that $\mathrm{cof}(\mathcal{N}) = \aleph_1$. Hence, it is consistent with ZFC that

$$\mathrm{cof}(\mathcal{N}) < \hat{\mathfrak{s}}. \tag{31}$$

Open problem. Is there any relation between
a) $\hat{\mathfrak{s}}$ and $\mathrm{cof}(\mathcal{M})$?
b) $\hat{\mathfrak{s}}$ and $\mathrm{non}(\mathcal{N})$?
c) $\hat{\mathfrak{s}}$ and $\mathrm{cof}(\mathcal{N})$?
d) $\hat{\mathfrak{s}}$ and $\mathfrak{d}$?

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ON $F(p,n)$-FIBONACCI BICOMPLEX NUMBERS

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Abstract

In this paper we introduce $F(p,n)$-Fibonacci bicomplex numbers and $L(p,n)$-Lucas bicomplex numbers as a special type of bicomplex numbers. We give some their properties and describe relations between them.

1. Introduction

Let consider the set $\mathbb{C}$ of complex numbers $a+bi$, where $a,b\in\mathbb{R}$, with the imaginary unit i. Let $\mathbb{B}$ be the set of bicomplex numbers w of the form

$$w = z_1 + z_2 j, \tag{1}$$

where $z_1, z_2 \in \mathbb{C}$. Then i and j are commuting imaginary units, i.e.

$$ij = ji,\ i^2 = j^2 = -1. \tag{2}$$

Let $w_1 = (a_1+b_1i)+(c_1+d_1i)j$ and $w_2 = (a_2+b_2i)+(c_2+d_2i)j$ be arbitrary two bicomplex numbers. Then the equality, the addition, the substraction, the multiplication and the multiplication by scalar are defined in the following way.
Equality: $w_1 = w_2$ only if $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$, $d_1 = d_2$,
addition: $w_1 + w_2 = ((a_1+a_2)+(b_1+b_2)i)+((c_1+c_2)+(d_1+d_2)i)j$,
substraction: $w_1 - w_2 = ((a_1-a_2)+(b_1-b_2)i)+((c_1-c_2)+(d_1-d_2)i)j$,
multiplication by scalar $s\in\mathbb{R}$: $sw_1 = (sa_1+sb_1i)+(sc_1+sd_1i)j$,
multiplication:
$$w_1\cdot w_2 = ((a_1a_2 - b_1b_2 - c_1c_2 + d_1d_2) + (a_1b_2 + a_2b_1 - c_1d_2 - c_2d_1)i) + \\ + ((a_1c_2 + a_2c_1 - b_1d_2 - b_2d_1) + (a_1d_2 + a_2d_1 + b_1c_2 + b_2c_1)i)\, j.$$

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The bicomplex numbers were introduced in 1892 by Segre, see [5]. The theory of bicomplex numbers is developed, many of papers concerning this topic are published quite recently, see for example [2], [3], [4].

The Fibonacci numbers F_n are defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$ with $F_0 = F_1 = 1$. The nth Lucas number L_n is defined recursively by $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$ with the initial terms $L_0 = 2$, $L_1 = 1$.

In this paper we recall some generalizations of Fibonacci numbers and Lucas numbers and we introduce the bicomplex numbers related with these generalizations.

2. The $F(p,n)$-Fibonacci numbers

The Fibonacci sequence has been generalized in many ways but a very natural is firstly to use one-parameter generalization of the Fibonacci sequence. A generalization uses one parameter p, $p \geq 2$ was introduced and studied by Kwaśnik and I. Włoch in the context of the number of p-independent sets in graphs, see [1]. We recall this definition.

Let $p \geq 2$ be integer. Then

$$\begin{aligned} &F(p,n) = n+1, \text{ for } n = 0,1,\ldots,p-1,\\ &F(p,n) = F(p,n-1) + F(p,n-p), \text{ for } n \geq p, \end{aligned} \tag{3}$$

is the $F(p,n)$-Fibonacci number.

Moreover $L(p,n)$-Lucas number is a cyclic version of $F(p,n)$ defined in the following way

$$\begin{aligned} &L(p,n) = n+1, \text{ for } n = 0,1,\ldots,2p-1,\\ &L(p,n) = L(p,n-1) + L(p,n-p), \text{ for } n \geq 2p, \end{aligned} \tag{4}$$

where $p \geq 2$, $n \geq 0$.
Note that for $n \geq 0$ we have that $F(2,n) = F_{n+1}$ and for $n \geq 2$ $L(2,n) = L_n$.

The following Tables present the initial words of the generalized Fibonacci numbers and the generalized Lucas numbers for special case of n and p.

n	0	1	2	3	4	5	6	7	8	9	10
F_n	1	1	2	3	5	8	13	21	34	55	89
$F(2,n)$	1	2	3	5	8	13	21	34	55	89	144
$F(3,n)$	1	2	3	4	6	9	13	19	28	41	60
$F(4,n)$	1	2	3	4	5	7	10	14	19	26	36
$F(5,n)$	1	2	3	4	5	6	8	11	15	20	26

Table 1. The values of $F(p,n)$ and F_n.

n	0	1	2	3	4	5	6	7	8	9	10
L_n	2	1	3	4	7	11	18	29	47	76	123
$L(2,n)$	1	2	3	4	7	11	18	29	47	76	123
$L(3,n)$	1	2	3	4	5	6	10	15	21	31	46
$L(4,n)$	1	2	3	4	5	6	7	8	13	19	26

Table 2. The values of $L(p,n)$ and L_n.

Generalized Fibonacci numbers $F(p,n)$ and generalized Lucas numbers $L(p,n)$ have been studied recently, mainly with respect to their graph and combinatorial properties, see for example [7], [8], [9], [10]. Among other some identities for $F(p,n)$ and $L(p,n)$ were given. We recall some of them.

Theorem 1 ([8]). *Let $p \geq 2$ be integer. Then for $n \geq p+1$*

$$\sum_{l=0}^{n-p} F(p,l) = F(p,n) - p. \tag{5}$$

Theorem 2 ([8]). *Let $p \geq 2$, $n \geq p$ be integers. Then*

$$\sum_{l=1}^{n} F(p,lp-1) + 1 = F(p,np). \tag{6}$$

Theorem 3 ([6]). *Let $p \geq 2$, $n \geq p$ be integers. Then*

$$\sum_{l=1}^{n} F(p,lp) = F(p,np+1) - F(p,1), \tag{7}$$

$$\sum_{l=1}^{n} F(p,lp+1) = F(p,np+2) - F(p,2), \tag{8}$$

$$\sum_{l=1}^{n} F(p,lp+2) = F(p,np+3) - F(p,3). \tag{9}$$

Theorem 4 ([8]). *Let $p \geq 2$, $n \geq 2p-2$ be integers. Then*

$$F(p,n) = \sum_{l=0}^{p-1} F(p,n-(p-1)-l). \tag{10}$$

Theorem 5 ([8]). *Let $p \geq 2$, $n \geq 2p$ be integers. Then*

$$\sum_{l=2}^{n} L(p,pl) = L(p,np+1) - (p+2). \tag{11}$$

Theorem 6 ([6]). *Let $p \geq 2$, $n \geq 2p$ be integers. Then*

$$\sum_{l=2}^{n} L(p, pl+1) = L(p, np+2) - L(p, p+2). \tag{12}$$

$$\sum_{l=2}^{n} L(p, pl+2) = L(p, np+3) - L(p, p+3). \tag{13}$$

$$\sum_{l=2}^{n} L(p, pl+3) = L(p, np+4) - L(p, p+4). \tag{14}$$

Theorem 7 ([8]). *Let $p \geq 2$, $n \geq 2p$ be integers. Then*

$$L(p, n) = pF(p, n-(2p-1)) + F(p, n-p). \tag{15}$$

3. The $F(p, n)$-Fibonacci bicomplex numbers

Let $n \geq 0$ be an integer. The nth $F(p, n)$-Fibonacci bicomplex number BF_n^p and the nth $L(p, n)$-Lucas bicomplex number BL_n^p are defined as

$$BF_n^p = (F(p, n) + F(p, n+1)i) + (F(p, n+2) + F(p, n+3)i)j, \tag{16}$$

$$BL_n^p = (L(p, n) + L(p, n+1)i) + (L(p, n+2) + L(p, n+3)i)j, \tag{17}$$

respectively.

Using the above definitions we can write selected $F(p, n)$-Fibonacci bicomplex numbers, i.e.

$$\begin{aligned}
&BF_0^3 = (1+2i) + (3+4i)j,\\
&BF_1^3 = (2+3i) + (4+6i)j,\\
&BF_2^3 = (3+4i) + (6+9i)j,\\
&\ldots\\
&BF_0^4 = (1+2i) + (3+4i)j,\\
&BF_1^4 = (2+3i) + (4+5i)j,\\
&BF_2^4 = (3+4i) + (5+7i)j,\\
&\ldots\\
&BF_0^5 = (1+2i) + (3+4i)j,\\
&BF_1^5 = (2+3i) + (4+5i)j,\\
&BF_2^5 = (3+4i) + (5+6i)j,\\
&\ldots
\end{aligned}$$

In the same way one can easily write selected $L(p, n)$-Lucas bicomplex numbers.

The addition, the subtraction and the multiplication of $F(p, n)$-Fibonacci bicomplex numbers and $L(p, n)$-Lucas bicomplex numbers are defined in the same way as for bicomplex numbers.

In the set $\mathbb{C}$, the complex conjugate of $x+yi$ is $\overline{x+yi}=x-yi$. In the set $\mathbb{B}$, for a bicomplex number $w=(a+bi)+(c+di)j$, there are three distinct conjugations.
Let BF_n^p be the nth $F(p,n)$-Fibonacci bicomplex number, i.e.
$BF_n^p=(F(p,n)+F(p,n+1)i)+(F(p,n+2)+F(p,n+3)i)j$,
The bicomplex conjugation of BF_n^p with respect to i has the form

$$\begin{aligned}\overline{BF_n^p}^i &= \overline{(F(p,n)+F(p,n+1)i)}+\overline{(F(p,n+2)+F(p,n+3)i)}j=\\ &= (F(p,n)-F(p,n+1)i)+(F(p,n+2)-F(p,n+3)i)j.\end{aligned}$$

The bicomplex conjugation of BF_n^p with respect to j has the form

$$\begin{aligned}\overline{BF_n^p}^j &= (F(p,n)+F(p,n+1)i)-(F(p,n+2)+F(p,n+3)i)j=\\ &= (F(p,n)+F(p,n+1)i)+(-F(p,n+2)-F(p,n+3)i)j.\end{aligned}$$

The third kind of conjugation is a composition of the above two conjugations. Putting $k:=ji=ij$ we can define the bicomplex conjugation of BF_n^p with respect to k as follows

$$\begin{aligned}\overline{BF_n^p}^k &= \overline{(F(p,n)+F(p,n+1)i)}-\overline{(F(p,n+2)+F(p,n+3)i)}j=\\ &= (F(p,n)-F(p,n+1)i)+(-F(p,n+2)+F(p,n+3)i)j.\end{aligned}$$

Using the bicomplex conjugation of BF_n^p with respect to i, j, k respectively and (16) we can write

$$\begin{aligned}&BF_n^p\cdot\overline{BF_n^p}^i=\\ &=\left(|F(p,n)+F(p,n+1)i|^2-|F(p,n+2)+F(p,n+3)i|^2\right)+\\ &\quad+2\Re\left((F(p,n)+F(p,n+1)i)\cdot\overline{(F(p,n+2)+F(p,n+3)i)}\right)j=\\ &=(F(p,n))^2+(F(p,n+1))^2-(F(p,n+2))^2-(F(p,n+3))^2+\\ &\quad+2\left(F(p,n)F(p,n+2)+F(p,n+1)F(p,n+3)\right)j.\end{aligned}$$

$$\begin{aligned}&BF_n^p\cdot\overline{BF_n^p}^j=\\ &=(F(p,n)+F(p,n+1)i)^2+(F(p,n+2)+F(p,n+3)i)^2=\\ &=(F(p,n))^2-(F(p,n+1))^2+(F(p,n+2))^2-(F(p,n+3))^2+\\ &\quad+2\left(F(p,n)F(p,n+1)+F(p,n+2)F(p,n+3)\right)i.\end{aligned}$$

$$\begin{aligned}&BF_n^p\cdot\overline{BF_n^p}^k=\\ &=\left(|F(p,n)+F(p,n+1)i|^2+|F(p,n+2)+F(p,n+3)i|^2\right)+\\ &\quad-2\Im\left((F(p,n)+F(p,n+1)i)\cdot\overline{(F(p,n+2)+F(p,n+3)i)}\right)k=\\ &=(F(p,n))^2+(F(p,n+1))^2+(F(p,n+2))^2+(F(p,n+3))^2+\\ &\quad-2\left(F(p,n+1)F(p,n+2)-F(p,n)F(p,n+3)\right)k.\end{aligned}$$

In the set $\mathbb{C}$, the modulus of $x+yi$ is $|x+yi|=\sqrt{(x+yi)\cdot\overline{(x+yi)}}=\sqrt{x^2+y^2}$. In the set $\mathbb{B}$ there are four different moduli, named: real modulus

$|BF_n^p|$, $i-$modulus $|BF_n^p|_i$, $j-$modulus $|BF_n^p|_j$ and $k-$modulus $|BF_n^p|_k$. We give the formulae of the squares of these modules:

$$|BF_n^p|^2 = |F(p,n) + F(p,n+1)i|^2 + |F(p,n+2) + F(p,n+3)i|^2 = \\ = (F(p,n))^2 + (F(p,n+1))^2 + (F(p,n+2))^2 + (F(p,n+3)^2,$$

$$|BF_n^p|_i^2 = BF_n^p \cdot \overline{BF_n^p}^i,$$

$$|BF_n^p|_j^2 = BF_n^p \cdot \overline{BF_n^p}^j,$$

$$|BF_n^p|_k^2 = BF_n^p \cdot \overline{BF_n^p}^k.$$

The different conjugations and squares of modules for $L(p,n)$-Lucas bicomplex number BL_n^p are presented as follows

$$\overline{BL_n^p}^i = (L(p,n) - L(p,n+1)i) + (L(p,n+2) - L(p,n+3)i)j,$$

$$\overline{BL_n^p}^j = (L(p,n) + L(p,n+1)i) + (-L(p,n+2) - L(p,n+3)i)j,$$

$$\overline{BL_n^p}^k = (L(p,n) - L(p,n+1)i) + (-L(p,n+2) + L(p,n+3)i)j.$$

$$|BL_n^p|^2 = (L(p,n))^2 + (L(p,n+1))^2 + (L(p,n+2))^2 + (L(p,n+3)^2,$$

$$|BL_n^p|_i^2 = (L(p,n))^2 + (L(p,n+1))^2 - (L(p,n+2))^2 - (L(p,n+3))^2 + \\ +2\left(L(p,n)L(p,n+2) + L(p,n+1)L(p,n+3)\right)j.$$

$$|BL_n^p|_j^2 = (L(p,n))^2 - (L(p,n+1))^2 + (L(p,n+2))^2 - (L(p,n+3))^2 + \\ +2\left(L(p,n)L(p,n+1) + L(p,n+2)L(p,n+3)\right)i.$$

$$|BL_n^p|_k^2 = (L(p,n))^2 + (L(p,n+1))^2 + (L(p,n+2))^2 + (L(p,n+3))^2 + \\ -2\left(L(p,n+1)L(p,n+2) - L(p,n)L(p,n+3)\right)k.$$

4. Properties of $F(p,n)$-Fibonacci bicomplex numbers

We will give some properties of $F(p,n)$-Fibonacci bicomplex numbers and $L(p,n)$-Lucas bicomplex numbers.

Theorem 8. *Let $p \geq 2$ be integer. Then for $n \geq p+1$*

(18)

$$\sum_{l=0}^{n-p} BF_l^p = BF_n^p - [p + (p + F(p,0))i + \\ + ((p + F(p,0) + F(p,1)) + (p + F(p,0) + F(p,1) + F(p,2))i)j].$$

Proof. Using (5) and (16) we have

$$\begin{aligned}
&\sum_{l=0}^{n-p} BF_l^p = BF_0^p + BF_1^p + \ldots + BF_{n-p}^p = \\
&= (F(p,0) + F(p,1)i) + (F(p,2) + F(p,3)i)j + \\
&+(F(p,1) + F(p,2)i) + (F(p,3) + F(p,4)i)j + \ldots + \\
&+(F(p,n-p) + F(p,n-p+1)i) + \\
&+(F(p,n-p+2) + F(p,n-p+3)i)j = \\
&= F(p,0) + F(p,1) + \ldots + F(p,n-p) + \\
&+ (F(p,1) + \ldots + F(p,n-p+1) + F(p,0) - F(p,0))\, i + \\
&+ [F(p,2) + \ldots + F(p,n-p+2) + F(p,0) + F(p,1) - F(p,0) + \\
&-F(p,1) + (F(p,3) + \ldots + F(p,n-p+3) + F(p,0) + F(p,1) + \\
&+F(p,2) - F(p,0) - F(p,1) - F(p,2))\, i]\, j = \\
&= (F(p,n) - p + (F(p,n+1) - p - F(p,0))i) + \\
&+ [(F(p,n+2) - p - F(p,0) - F(p,1)) + \\
&+ (F(p,n+3) - p - F(p,0) - F(p,1) - F(p,2))\, i]\, j = \\
&= BF_n^p - (p + (p + F(p,0))\, i) - [(p + F(p,0) + F(p,1)) + \\
&+ (p + F(p,0) + F(p,1) + F(p,2))\, i]\, j,
\end{aligned}$$

which ends the proof. □

Theorem 9. *Let $p \geq 2$, $n \geq p$ be integers. Then*

$$\sum_{l=1}^{n} BF_{lp-1}^p = BF_{np}^p - [(F(p,0) + F(p,1)i) + (F(p,2) + F(p,3)i)\, j]\,. \tag{19}$$

Proof. Using (16) we have

$$\begin{aligned}
&\sum_{l=1}^{n} BF_{lp-1}^p = BF_{p-1}^p + BF_{2p-1}^p + \ldots + BF_{np-1}^p = \\
&= (F(p,p-1) + F(p,p)i) + (F(p,p+1) + F(p,p+2)i)j + \\
&+(F(p,2p-1) + F(p,2p)i) + (F(p,2p+1) + F(p,2p+2)i)j + \ldots + \\
&+(F(p,np-1) + F(p,np)i) + (F(p,np+1) + F(p,np+2)i)j = \\
&= F(p,p-1) + F(p,2p-1) + \ldots + F(p,np-1) + \\
&+ (F(p,p) + F(p,2p) + \ldots + F(p,np))\, i + \\
&+ [(F(p,p+1) + F(p,2p+1) + \ldots + F(p,np+1)) + \\
&+ (F(p,p+2) + F(p,2p+2) + \ldots + F(p,np+2))\, i]\, j.
\end{aligned}$$

Writing (6) as $\sum_{l=1}^{n} F(p,lp-1) = F(p,np) - 1 = F(p,np) - F(p,0)$ and using (7)–(9) we obtain (19). □

Theorem 10. *Let $p \geq 2$, $n \geq 2p-2$ be integers. Then*

$$BF_n^p = \sum_{l=0}^{p-1} BF_{n-(p-1)-l}^p. \tag{20}$$

Proof. Using (10) and (16) we have

$$\begin{aligned}
&\sum_{l=0}^{p-1} BF^p_{n-(p-1)-l} = BF^p_{n-(p-1)} + BF^p_{n-(p-1)-1} + \ldots + BF^p_{n-(p-1)-(p-1)} = \\
&= (F(p, n-(p-1)) + F(p, n-(p-1)+1)i) + \\
&+ [F(p, n-(p-1)+2) + F(p, n-(p-1)+3)i]\, j + \\
&+ (F(p, n-(p-1)-1) + F(p, n-(p-1))i) + \\
&+ [F(p, n-(p-1)+1) + F(p, n-(p-1)+2)i]\, j + \ldots + \\
&+ (F(p, n-(p-1)-(p-1)) + F(p, n-(p-1)-(p-1)+1)i) + \\
&+ [F(p, n-(p-1)-(p-1)+2) + F(p, n-(p-1)-(p-1)+3)i]\, j = \\
&= (F(p,n) + F(p,n+1)i) + (F(p,n+2) + F(p,n+3)i)j = BF^p_n,
\end{aligned}$$

which ends the proof. □

Theorem 11. *Let $p \geq 2$, $n \geq 2p$ be integers. Then*

$$\sum_{l=2}^{n} BL^p_{pl} = BL^p_{np+1} - BL^p_{p+1}. \tag{21}$$

Proof. Using (17) we have

$$\begin{aligned}
&\sum_{l=2}^{n} BL^p_{pl} = BL^p_{2p} + BL^p_{3l} + \ldots + BL^p_{nl} = \\
&= (L(p,2p) + L(p,2p+1)i) + (L(p,2p+2) + L(p,2p+3)i)j + \\
&+ (L(p,3p) + L(p,3p+1)i) + (L(p,3p+2) + L(p,3p+3)i)j + \ldots + \\
&+ (L(p,np) + L(p,np+1)i) + (L(p,np+2) + L(p,np+3)i)j + \\
&= L(p,2p) + L(p,3p) + \ldots + L(p,np) + \\
&+ (L(p,2p+1) + L(p,3p+1) + \ldots + L(p,np+1))\, i + \\
&+ [(L(p,2p+2) + L(p,3p+2) + \ldots + L(p,np+2)) + \\
&+ (L(p,2p+3) + L(p,3p+3) + \ldots + L(p,np+3))\, i]\, j.
\end{aligned}$$

Writing (11) as $\sum_{l=2}^{n} L(p,pl) = L(p,np+1) - L(p,p+1)$ and using (12)–(14) we obtain (21). □

Theorem 12. *Let $p \geq 2$, $n \geq 2p$ be integers. Then*

$$BL^p_n = p \cdot BF^p_{n-(2p-1)} + BF^p_{n-p}. \tag{22}$$

Proof. Using (16) we have

$$\begin{aligned}
BF^p_{n-(2p-1)} &= (F(p, n-(2p-1)) + F(p, n-(2p-1)+1)i) + \\
&+ (F(p, n-(2p-1)+2) + F(p, n-(2p-1)+3)i)j
\end{aligned}$$

and

$$\begin{aligned}
BF^p_{n-p} &= (F(p, n-p) + F(p, n-p+1)i) + \\
&+ (F(p, n-p+2) + F(p, n-p+3)i)j,
\end{aligned}$$

consequently

$$\begin{aligned}
&p \cdot BF^p_{n-(2p-1)} + BF^p_{n-p} = \\
&= p \cdot F(p, n-(2p-1)) + F(p, n-p)+ \\
&+ (p \cdot F(p,(n+1)-(2p-1)) + F(p,(n+1)-p))\, i+ \\
&+ [(p \cdot F(p,(n+2)-(2p-1)) + F(p,(n+2)-p)) + \\
&+ (p \cdot F(p,(n+3)-(2p-1)) + F(p,(n+3)-p))\, i]\, j
\end{aligned}$$

Using (15) we have

$$\begin{aligned}
&p \cdot BF^p_{n-(2p-1)} + BF^p_{n-p} = \\
&= (L(p,n) + L(p,n+1)i) + (L(p,n+2) + L(p,n+3)i)j,
\end{aligned}$$

which ends the proof. □

For integers p, n, l, $p \geq 2$, $n \geq 2$, $0 \leq l \leq n$ we have (see [9]) the direct formula for $F(p,n)$-Fibonacci number

$$F(p,n) = \sum_{l \geq 0} f(p,n,l),$$

where

$$f(p,n,l) = \binom{n-(p-1)(l-1)}{l}.$$

Using this direct formula other forms of given earlier identities can be given.

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SANGAKU FAN SHAPE PROBLEMS

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Abstract

The paper discuss three sangaku problems on relationship among circles inscribed in the sector of an annulus, which is due to its shape, called a fan.

1. Introduction

In the period 17^{th} to 19^{th} century, so called Edo period when Japan closed its doors to the outer world, traditional Japanese mathematics (*wasan*), was developed. In Japan in that times, there was no official academia, so mathematics was developed not only by scholars but also by mathematical laity, that had found mathematics divine. Mathematics enthusiasts dedicated to shrines and temples the wooden tablets on which mathematics problems were written. Those votive tablets are called sangaku. The problems featured on the sangaku are typical problems of japanese mathematics (wasan) and often involve many circles which is uncommon in western mathematics. Each tablet states a theorem or a problem. It is a invitation and a challenge to other experts to prove the theorem or to solve the problem. Most sangaku contain only the final answer to a problem, rarely a detailed solution. It is a work of art as well as a mathematical statement. Sangaku are perishable, and the majority of them have decayed and disappeared during the last two centuries.

2. Main results

The first problem can be found on the top right corner of the Katayamahiko shrine tablet (Fukagawa, Rothman, 2008).

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PROBLEM 1. *As shown in Figure 1, in a sector of annulus of radius R, two circles of radius r are tangent to each other and touch the sector internally. A small circle of radius t touches both the sector and a chord of length d. If $d = 3.62438$ and $2t = 0.34$, find $2r$.*
Tablet contains an answer: $2r = 3,025$.

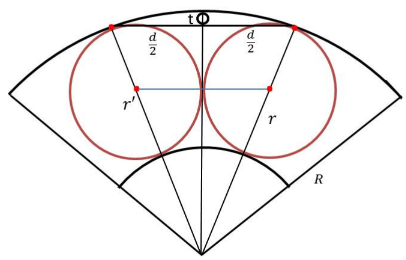

FIGURE 1

Solution. Let O be center of concentric circles that determine the circular ring, C_1 and C_2 two equal circles inscribed in circular ring. Denote by A, B and C the points at which the observed circles touch the edge of the circular ring, with F the touch point of two equal circles, and with D the point where the chord AB touches the small circle (Figure 2).

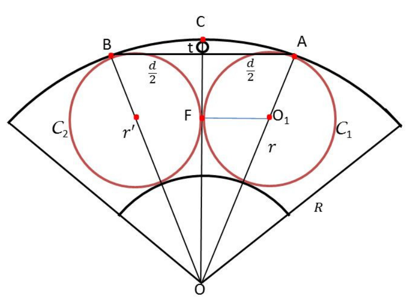

FIGURE 2

Applying the Pythagorean theorem on the rectangular triangle $\triangle ODA$ gives

$$R^2 = (R - 2t)^2 + \left(\frac{d}{2}\right)^2$$

which implies

$$R = \frac{d^2}{16t} + t \tag{1}$$

Similarly, using the Pythagorean theorem to $\triangle O_1FO$ (where O_1 is the center of circle C_1), we obtain

$$(R - r)^2 = r^2 + (R - r - 2t)^2$$

whence it follow

$$R = r + t + \frac{r^2}{4t} \tag{2}$$

Equating the expressions (1) and (2) for R and solving the resulting quadratic for r gives

$$r = \sqrt{4t^2 + \frac{d^2}{4}} - 2t$$

If $d = 3,62438$ and $2t = 0,34$, then $r = 1,5038$ or $2r = 3,0076$, which is a slightly different result from the one on the Katayamahiko shrine tablet.

The next, central problem in this paper, is given on sangaku in the temple Isaniwa in Ehime Prefecture, well known for its 22 tablets preserved to present day. Tablet (Figure 3) is dated in 1873 (Syomin-no-sanjyutsuten, 2005).

PROBLEM 2. *Let fan makes a third of an annulus, within which one inscribed seven circles: one eastern, two western, two southern and two northern circles. If the diameter of the southern circles given, what is the diameter of the northern circle?*

FIGURE 3

Solution. Denote by C_1 eastern circle, and by C_2, C_3 and C_4 western, southern and northern circle respectively. Let O_1, O_2, O_3, O_4 are centers of circles C_1, C_2, C_3 and C_4 respectively, and r_1, r_2, r_3, r_4 their radii. Let R is the radius of the annulus outer circle and O its center. Let us introduce other symbols as in Figure 4.

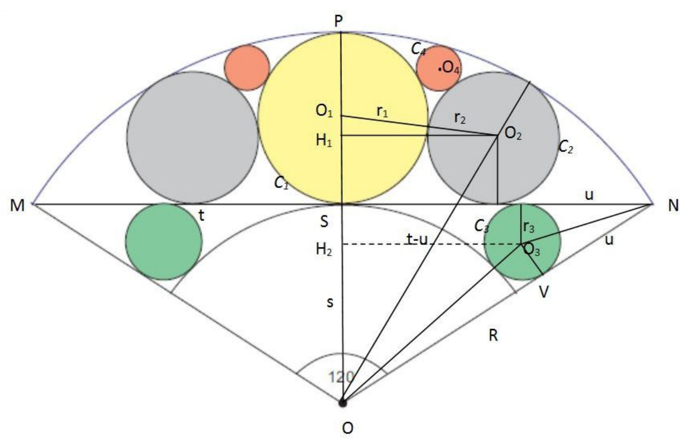

FIGURE 4

In $\triangle OSN$ there are $\angle SON = 60°$ and $\angle OSN = 90°$. Hence,

$$|OS| = s = R \cdot \cos 60° = \frac{R}{2}$$

$$|SN| = t = R \cdot \sin 60° = \frac{R\sqrt{3}}{2}$$

Radius r_1 of the circle C_1 is obtained as follows:

$$|OP| = R \qquad \text{and} \qquad |OP| = |OS| + |SP|$$

i.e.

$$R = s + 2r_1$$

from which we obtain

$$r_1 = \frac{R - s}{2} = \frac{R}{4}.$$

Notice rectangular triangles $\triangle OH_1O_2$ and $\triangle O_1H_1O_2$. In $\triangle OH_1O_2$ there is

$$|H_1O_2|^2 = |OO_2|^2 - |OH_1|^2 \tag{3}$$

and in $\triangle O_1H_1O_2$

$$|H_1O_2|^2 = |O_2O_1|^2 - |O_1H_1|^2. \tag{4}$$

From (3) and (4) we get

$$|OO_2|^2 - |OH_1|^2 = |O_2O_1|^2 - |O_1H_1|^2,$$

respectively

$$(R - r_2)^2 - \left(\frac{R}{2} + r_2\right)^2 = \left(\frac{R}{4} + r_2\right)^2 - \left(\frac{R}{4} - r_2\right)^2.$$

Radius r_2 of the circle C_2 can be expressed using previous equality:

$$r_2 = \frac{3r}{16}. \tag{5}$$

Applying Pythagorean theorem on triangles $\triangle OH_2O_3$ and $\triangle OO_3V$ we obtain

$$|OO_3|^2 = |OH_2|^2 + |H_2O_3|^2$$

and introducing notation $u = |NV|$, previous equality becomes

$$\left(\frac{R}{2} + r_3\right)^2 = \left(\frac{R}{2} - r_3\right)^2 + (t - u)^2.$$

Rearranging the last equality and taking into account (2) we obtain

$$\frac{R\sqrt{3}}{2} - u = \sqrt{2Rr_3}. \tag{6}$$

In $\triangle OO_3V$ there is

$$|OO_3|^2 = |OV|^2 + |VO_3|^2,$$

respectively

$$\left(\frac{R}{2} + r_3\right)^2 = r_3^2 + (R-u)^2 .$$

Finally,

$$\frac{R^2}{4} + Rr_3 = (R-u)^2. \tag{7}$$

Radius r_3 of the circle C_3 and segment $u = |NV|$ can be expressed in terms of R using (6) and (7)

$$r_3 = \frac{3\left(2-\sqrt{3}\right)R}{2\left(2+\sqrt{3}\right)} \cdot u = \frac{3R}{2\left(2+\sqrt{3}\right)}. \tag{8}$$

Lastly, to determine the required radius r_4 of northern circle, we will introduce notations:

$$|O_1H_3| = p,$$

$$|O_4H_3| = q,$$

$$|O_2H_4| = z.$$

In $\triangle OH_3O_4$ there is

$$(R-r_4)^2 = \left(\frac{R}{2} + r_1 + p\right)^2 + q^2$$

and in $\triangle O_1O_2H_1$ there is

$$(r_1+r_2)^2 = (r_1-r_2)^2 + (q+z)^2 ,$$

wheres

$$q + z = 2\sqrt{r_1r_2}.$$

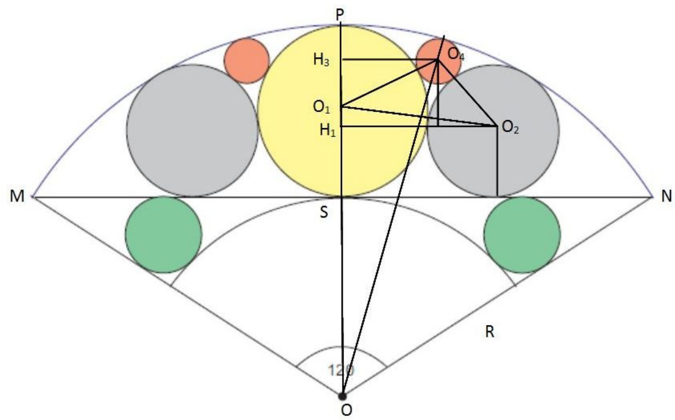

FIGURE 5

Applying the Pythagorean theorem on the rectangular triangles $\triangle O_4H_4O_2$ and $\triangle O_1H_3O_4$ respectively (Figure 5), we obtain

$$(r_3 + r_4)^2 = (p + r_1 - r_2)^2 + z^2,$$
$$p^2 + q^2 = (r_1 + r_4)^2 .$$

Segments p, q, z, r_4 that appeared in previous equalities can be expressed in term if R:

$$r_4 = \frac{3}{193}\left(25 - 12\sqrt{3}\right)R,$$

$$p = \frac{1}{772}\left(-307 + 240\sqrt{3}\right)R,$$

$$q = \frac{2}{193}\left(3 + 14\sqrt{3}\right)R,$$

$$z = \frac{3}{772}\left(-8 + 27\sqrt{3}\right)R.$$

Finally, observing the ratio of radii of circles C_4 and C_3 gives

$$\frac{r_4}{r_3} = \frac{\frac{3}{193}\left(25-12\sqrt{3}\right)R}{\frac{3\left(2-\sqrt{3}\right)R}{2\left(2+\sqrt{3}\right)}} = \cdots = \frac{62+\sqrt{3\cdot 1024}}{193}$$

and

$$r_4 = r_3 \cdot \frac{62+\sqrt{3\cdot 1024}}{193}. \tag{9}$$

Equality (9) corresponds to the solution stated on sangaku in temple Isaniwa.

The third problem dates back to 1865. and is given on sangaku in Meiseirinji temple (Syomin-no-sanjyutsuten, 2005). In solution of this problem inversion technique will be used. Theorem of inversion of circles will be stated without proof.

Theorem 1. *A circle, its inverse, and the center of inversion are collinear.*

PROBLEM 3. *Inside a fan-shaped sector five circles touch each other; one is a "red" circle of radius r_1, two are "green" circles of radius r_2, and two are "white" circles of radius r_3. The radius of the sector is r, and the circles touch each other symmetrically about the center O. We take the angle of the sector to be variable and r constant. As the angle is varied, the inner radius of the sector t is adjusted so that the five circles continue to touch; r_3 is also allowed to vary, while the other radii remain constant.*
Show that $2\left(r_1+r_3\right)=r$, when r_3 is a maximum.

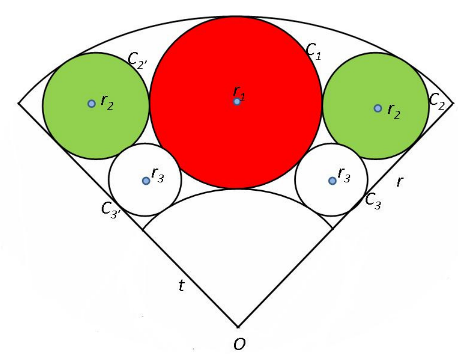

FIGURE 6

Solution. Let C_1 denote "red" circle, C_2 and $C_{2'}$ two "green" circles and C_3 and $C_{3'}$ two "white" circles, as in Figure 6. Under conditions of the problem, it is sufficient to consider one half of the figure given.
Assume initially that the center O and the centers O_2 and O_3 of the circles C_2 and C_3 are collinear. Then, Figure 7 shows that

$$r = t + 2r_1 = t + 2r_3 + 2r_2 \tag{10}$$

or

$$r_1 = r_2 + r_3. \tag{11}$$

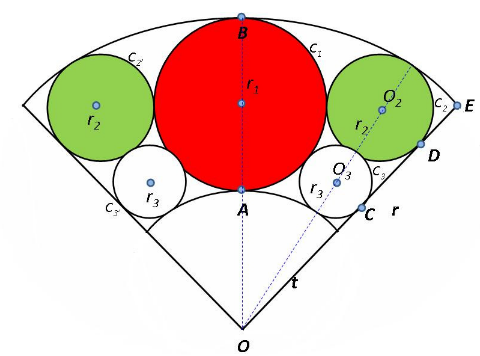

FIGURE 7

Similar triangles $\triangle OO_3C$ and $\triangle OO_2D$ give

$$\frac{r_3}{t+r_3} = \frac{r_2}{t+2r_3+r_2}.$$

Eliminating t by previous equality and using (10) give

$$\frac{r_3}{r-2r_2-r_3} = \frac{r_2}{r-r_2}$$

or

$$r_3 = \frac{1}{r}\left(-2r_2^2 + rr_2\right). \tag{12}$$

Expression (12) can be rewritten as

$$r_3 = \frac{1}{r}\left[-2\left(r_2 - \frac{r}{4}\right)^2 + \frac{r^2}{8}\right].$$

Given condition of constant radius r, last expression implies that r_3 is maximized and equals $\frac{r}{8}$ when $r_2 = \frac{r}{4}$. This and (11) imply

$$2r_1 + 2r_3 = 2r_2 + 4r_3 = r.$$

Therefore, the statement is proven in case of collinear centers of circles with radii r, r_2 and r_3.
It remains to prove that the aforementioned centers of circles collinear. In this purpose, consider a Figure 7 and make use of Theorem 1. Choosing O as the center of inversion, if we can invert r_2 into r_3 and vice versa, we have shown that the two circles are collinear with O, and the rest of the proof follows.
To do this, notice that if in Figure 7 we invert circle with radius t into circle with radius r, and vice versa, then circle with radius r_1 must invert into itself in order to keep the points of tangency A and B invariant. Similarly, circles with radii r_3 and r_2 are tangent to circle with radius r_1 and to the line OE at the points C and D. In order that all points of tangency are preserved, in particular that C inverts into D and vice versa, then circle with radius r_2 must invert into circle with radius r_3, and the reverse. To do this, merely choose the radius of inversion k such that $k^2 = rt$.

3. Final remarks

In the Edo era of the 18^{th} and 19^{th} centuries in Japan, ordinary people enjoyed mathematics in daily life, not as a professional study but rather as an intellectual popular game and a recreational activity. Sangaku usually don't provide a proof of the theorem, and even books of them have been published in Japan for many years, some theorems are still unsolved. It gives opportunity to researchers to explore and decrypt sangaku problems as well as to link similar problems. Sangaku can be used to stimulate the interest of students in mathematics as many of sangaku problems are a source of pleasure and challenge.

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SOME REMARKS ABOUT K-CONTINUITY OF K-SUPERQUADRATIC MULTIFUNCTIONS

KATARZYNA TROCZKA-PAWELEC

Abstract

Let $X = (X, +)$ be an arbitrary topological group. The set-valued function $F\colon X \to n(Y)$ is called K-superquadratic iff

$$F(x+y) + F(x-y) \subset 2F(x) + 2F(y) + K,$$

for all $x, y \in X$, where Y denotes a topological vector space and K is a cone.

In this paper the $K-$continuity problem of multifunctions of this kind will be considered with respect to $K-$boundedness. The case where $Y = \mathbb{R}^N$ will be considered separately.

1. Introduction

Let $X = (X, +)$ be an arbitrary topological group. A real-valued function f is called superquadratic, if it fulfils inequality

$$2f(x) + 2f(y) \leq f(x+y) + f(x-y), \qquad x, y \in X. \tag{1}$$

If the sign " $\leq$ " in (1) is replaced by " $\geq$ ", then f is called subquadratic. The continuity problem of functions of this kind was considered in [2]. This problem was also considered in the class of set-valued functions. By the set-valued functions we understand functions of the type $F\colon X \to 2^Y$, where X and Y are given sets. Throughout this paper set-valued functions will be always denoted by capital letters. A set-valued function F is called superquadratic if it satisfies inclusion

$$2F(x) + 2F(y) \subset F(x+y) + F(x-y), \qquad x, y \in X, \tag{2}$$

and subquadratic set-valued function, if it satisfies inclusion defined in this form

$$F(x+y) + F(x-y) \subset 2F(x) + 2F(y), \qquad x, y \in X. \tag{3}$$

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For single-valued real functions properties of subquadratic and superquadratic functions are quite analogous and, in view of the fact that if a function f is subquadratic, then the function $-f$ is superquadratic and conversely, it is not necessary to investigate functions of these two kinds individually. In the case of set-valued functions the situation is different. Even if properties of subquadratic and superquadratic set-valued functions are similar, we have to proved them separately. If the sign " $\subset$ " in the inclusions above is replaced by " $=$ ", then F is called quadratic set-valued function. The class of quadratic set-valued functions is an important subclass of the class of subquadratic and superquadratic set-valued functions. Quadratic set-valued functions have already extensive bibliography (see W. Smajdor [5], D. Henney [1] and K. Nikodem [4]). The continuity problem of subquadratic and superquadratic set-valued functions was considered in [6] and [7].

Adding a cone K in the space of values of a set-valued function F lets us consider a K-superquadratic set-valued function , that is solution of the inclusion

$$F(x+y)+F(x-y)\subset 2F(x)+2F(y)+K, \qquad x,y\in X. \tag{4}$$

The concept of K-superquadraticity is related to real-valued superquadratic functions. Note, in the case when F is a single-valued real function and $K=[0,\infty)$, we obtain the standard definition of superquadratic functionals (1). Similarly, if a set-valued function F satisfies the following inclusion

$$2F(x)+2F(y)\subset F(x+y)+F(x-y)+K, \qquad x,y\in X \tag{5}$$

then it is called K-subquadratic. The K-continuity problem of multifunction of this kind was considered in [9]. In this paper we will consider the K-continuity problem for K-superquadratic set-valued functions. Likewise as in functional analysis we can look for connections between K-boundedness and K-semicontinuity of set-valued functions of this kind.

Assuming $K=\{0\}$ in (4) and (5) we obtain the inclusions (2) and (3).

Let us start with the notations used in this paper. Let Y be a topological vector space. We consider the family $n(Y)$ of all non-empty subsets of as a topological space with the Hausdorff topology. In this topology the set

$$N_W(A):=\{B\in n(Y): A\subset B+W, B\subset A+W\}$$

where W runs the base of neighbourhoods of zero in Y, form a base of neighbourhoods of a set $A\in n(Y)$. By $cc(Y)$ we denote the family of all compact and convex members of $n(Y)$. The term set-valued function will be abbreviated to the form s.v.f.

Now we present here some definitions for the sake of completeness. Recall that a set $K\subset Y$ is called a cone iff $K+K\subset K$ and $sK\subset K$ for all $s\in(0,\infty)$.

Definition 1. (cf. [3]) *A cone K in a topological vector space Y is said to be a normal cone iff there exists a base $\mathfrak{W}$ of zero in Y such that*

$$W = (W + K) \cap (W - K)$$

for all $W \in \mathfrak{W}$.

Definition 2. (cf. [3]) *An s.v.f. $F\colon X \to n(Y)$ is said to be K-upper semi-continuous (abbreviated $K-$u.s.c.) at $x_0 \in X$ iff for every neighbourhood V of zero in Y there exists a neighbourhood U of zero in X such that*

$$F(x) \subset F(x_0) + V + K$$

for every $x \in x_0 + U$.

Definition 3. (cf. [3]) *An s.v.f. $F\colon X \to n(Y)$ is said to be K-lower semi-continuous (abbreviated $K-$l.s.c.) at $x_0 \in X$ iff for every neighbourhood V of zero in Y there exists a neighbourhood U of zero in X such that*

$$F(x_0) \subset F(x) + V + K$$

for every $x \in x_0 + U$.

Definition 4. (cf. [3]) *An s.v.f. $F\colon X \to n(Y)$ is said to be $K-$continuous at $x_0 \in X$ iff it is both $K-$u.s.c. and $K-$l.s.c. at x_0. It is said to be $K-$continuous iff it is $K-$continuous at each point of X.*

Note that in the case where $K = \{0\}$ the $K-$continuity of F means its continuity with respect to the Hausdorff topology on $n(Y)$.

In the proof of the main theorems we will use some known lemmas (see Lemma 1.1, Lemma 1.3, Lemma 1.6 and Lemma 1.9 in [3]). The first lemma says that for a convex subset A of an arbitrary real vector space Y the equality $(s + t)A = sA + tA$ holds for every $s, t \geq 0$ or (s,t<0). The second lemma says that in a real vector space Y for two convex subsets A, B the set $A + B$ is also convex. The next lemma says that if $A \subset Y$ is a closed set and $B \subset Y$ is a compact set, where Y denotes a real topological vector space, then the set $A + B$ is closed. For any sets $A, B \subset Y$, where Y denotes the same space as above, the inclusion $\overline{A} + \overline{B} \subset \overline{A + B}$ holds and equality holds if and only if the set $\overline{A} + \overline{B}$ is closed.

Let us adopt the following three definitions which are natural extension of the concept of the boundedness for real-valued functions.

Definition 5. *An s.v. f. $F\colon X \to n(Y)$ is said to be $K-$lower bounded on a set $A \subset X$ iff there exists a bounded set $B \subset Y$ such that $F(x) \subset B + K$ for all $x \in A$. An s.v. f. $F\colon X \to n(Y)$ is said to be $K-$lower bounded at a point $x \in X$ iff there exists a neighbourhood U_x of zero in X such that F is $K-$lower bounded on a set $x + U_x$*

Definition 6. *An s.v. f. $F\colon X \to n(Y)$ is said to be $K-$upper bounded on a set $A \subset X$ iff there exists a bounded set $B \subset Y$ such that $F(x) \subset B - K$ for all $x \in A$. An s.v. f. $F\colon X \to n(Y)$ is said to be $K-$upper bounded at a point $x \in X$ iff there exists a neighbourhood U_x of zero in X such that F is $K-$upper bounded on a set $x + U_x$*

Definition 7. *An s.v. function $F\colon X \to n(Y)$ is said to be locally K-lower (upper) bounded in X if for every $x \in X$ there exists a neighbourhood U_x of zero in X such that F is K-lower (upper) bounded on a set $x + U_x$. It is said to be locally K-bounded in X if it is both locally K-lower and locally K-upper bounded in X.*

Definition 8. *We say that 2-divisible topological group X has the property $(\frac{1}{2})$ iff for every neighbourhood V of zero there exists a neighbourhood W of zero such that $\frac{1}{2}W \subset W \subset V$.*

For the $K-$superquadratic set-valued functions the following two theorems hold.

Theorem 1. (cf. [8]) *Let X be a 2-divisible topological group with property $(\frac{1}{2})$, Y locally convex topological real vector space and $K \subset Y$ a closed normal cone. If a K-superquadratic s.v.f. $F\colon X \to cc(Y)$ is K-u.s.c. at zero, $F(0) = \{0\}$ and locally K- bounded in X, then it is K-u.s.c. in X.*

Theorem 2. (cf. [10]) *Let X be a 2-divisible topological group, Y locally convex topological real vector space and $K \subset Y$ a closed normal cone. If a K-superquadratic s.v.f. $F\colon X \to cc(Y)$ is K-u.s.c. at zero, $F(0) = \{0\}$ and locally K- bounded in X then it is K-l.s.c. in X.*

Let us note, that Theorem 1 and Theorem 2, by Definition 4, yield directly the following main theorem for K-superquadratic multifunctions.

Theorem 3. *Let X be a 2-divisible topological group with property $(\frac{1}{2})$, Y locally convex topological real vector space and $K \subset Y$ a closed normal cone. If a K-superquadratic s.v.f. $F\colon X \to cc(Y)$ is K-u.s.c. at zero, $F(0) = \{0\}$ and locally K- bounded in X, then it is K-continuous in X.*

Let us introduce the following definitions.

Definition 9. *An s.v. f. $F\colon X \to n(Y)$ is said to be weakly $K-$lower bounded on a set $A \subset X$ iff there exists a bounded set $B \subset Y$ such that $F(x) \bigcap (B + K) \neq \emptyset$ for all $x \in A$.*

Definition 10. *An s.v. f. $F\colon X \to n(Y)$ is said to be weakly $K-$upper bounded on a set $A \subset X$ iff there exists a bounded set $B \subset Y$ such that $F(x) \bigcap (B - K) \neq \emptyset$ for all $x \in A$.*

Definition 11. *An s.v. f.* $F: X \to n(Y)$ *is said to be locally weakly* $K-$*upper bounded in* X *iff for every* $x \in X$ *there exists a neighbourhood* U_x *of zero in* X *such that* F *is* $K-$*upper bounded on a set* $x + U_x$.

Definition 12. *An s.v. f.* $F: X \to n(Y)$ *is said to be locally weakly* $K-$*lower bounded in* X *iff for every* $x \in X$ *there exists a neighbourhood* U_x *of zero in* X *such that* F *is* $K-$*lower bounded on a set* $x + U_x$.

Definition 13. *An s.v. f.* $F: X \to n(Y)$ *is said to be locally weakly* $K-$*bounded in* X *iff for every* $x \in X$ *there exists a neighbourhood* U_x *of zero in* X *such that* F *is weakly* $K-$*lower and weakly* $K-$*upper bounded on a set* $x + U_x$.

Clearly, if F is $K-$upper ($K-$lower) bounded on a set A, then it is weakly $K-$upper ($K-$lower) bounded on a set A. In the case of single-valued functions these definitions coincide.

For the $K-$superquadratic set-valued functions the following lemma holds.

Lemma 1. *Let* X *be a* $2-$*divisible topological group satisfying condition* $\left(\frac{1}{2}\right)$, Y *topological vector space and* $K \subset Y$ *a cone. Let* $F: X \to B(Y)$ *be a* $K-$*superquadratic s.v.f. , such that* $F(0) = \{0\}$ *and* $G: X \to n(Y)$ *be an s.v.f. with*

$$G(x) \subset F(x) + K \tag{6}$$

for all $x \in X$.

If F *is* $K-$*lower bounded at zero and* G *is locally weakly* K*-upper bounded in* X *, then* F *is locally* $K-$*lower bounded in* X.

Proof. Let $x \in X$. There exist a bounded set $B_1 \subset Y$ and a symmetric neighbourhood U_1 of zero in X such that

$$G(x - t) \cap (B_1 - K) \neq \emptyset, \quad t \in U_1,$$

which implies that that for all $t \in U_1$ there exists $a \in G(x - t)$ and $a \in (B_1 - K)$. Consequently, we get

$$0 = a - a \in G(x - t) - B_1 + K \tag{7}$$

for all $t \in U_1$. Since F is $K-$lower bounded at zero, there exist a symmetric neighbourhood U_2 of zero in X and a bounded set $B_2 \subset Y$ such that

$$F(t) \subset B_2 + K, \quad t \in U_2. \tag{8}$$

Let $\widetilde{U}$ be a symmetric neighbourhood of zero in X with $\frac{1}{2}\widetilde{U} \subset \widetilde{U} \subset U_1 \cap U_2$. Let $t \in \frac{1}{2}\widetilde{U}$. Using (6), (7) i (8), we obtain

$F(x+t)+0 \subset F(x+t)+G(x-t)-B_1+K \subset F(x+t)+F(x-t)-B_1+K \subset$

$$\subset 2F(x) + 2F(t) - B_1 + K \subset 2F(x) + 2B_2 - B_1 + K.$$

Define $\widetilde{B} := 2F(x) + 2B_2 - B_1$. Since $F(x)$ is a bounded set, then the set $\widetilde{B}$ is also bounded as the sum of bounded sets. Therefore

$$F(x+t) \subset \widetilde{B} + K, \quad t \in \frac{1}{2}\widetilde{U},$$

which means that F is locally $K-$lower bounded in X. $\square$

In the case of K-superquadratic multifunctions we require Y space to be locally bounded topological vector space. Then the following theorem holds.

Theorem 4. *Let X be a 2-divisible topological group with property $(\frac{1}{2})$, Y locally convex topological vector space and $K \subset Y$ a closed normal cone. If a K-superquadratic s.v.f. $F\colon X \to cc(Y)$ is K-u.s.c. at zero, $F(0) = \{0\}$ and locally K- upper bounded in X, then it is K-continous in X.*

Proof. Let W be a bounded neighbourhood of zero in Y. Since F is K-u.s.c. at zero and $F(0) = \{0\}$, then there exists a neighbourhood U of zero in X such that

$$F(t) \subset V + K$$

for all $t \in U$, which means that F is K-lower bounded at zero. The condition of locally K-upper boundedness in X implies F is locally K-weakly upper bounded in X. By Lemma 1 $(G = F)$ F is locally K-lower bounded in X. Consequently by Theorem 3 F is $K-$continuous at each point of X. $\square$

2. The case $n(\mathbb{R}^N)$

Now we consider the case where the space of values is $n(\mathbb{R}^N)$. In our next proof, we will use known following lemma.

Lemma 2. (cf. [9]) *Let Y be a topological vector space and K be a cone in Y. Let A, B, C be non-empty subsets of Y such that $A + C \subset B + C + K$. If B is convex and C is bounded then $A \subset \overline{B + K}$.*

For the $K-$superquadratic set-valued functions the following lemma holds.

Lemma 3. *Let X be a topological group and K a closed cone in $\mathbb{R}^N$. Let $F\colon X \to cc(\mathbb{R}^N)$ be a $K-$superquadratic s.v.f. with $F(0) = \{0\}$. If F is K-l.s.c. at some point $x_0 \in X$, then it is K-l.s.c. at zero.*

Proof. Let W be a neighbourhood of zero in Y.There exists a convex neighbourhood V of zero in Y such that the set $\overline{V}$ is compact with $3\overline{V} \subset W$. Since F is K-l.s.c. at $x_0 \in X$ then there exists a symmetric neighbourhood U of zero in X such that

$$F(x_0) \subset F(x_0 + t) + V + K, \tag{9}$$

$$F(x_0) \subset F(x_0 - t) + V + K, \tag{10}$$

for all $t \in U$.
Let $t \in U$. By convexity of the set $F(x_0)$ and by (9) i (10), we obtain

$$2F(x_0) \subset F(x_0 + t) + F(x_0 - t) + 2V + K \subset 2F(x_0) + 2F(t) + 2V + K.$$

Then

$$F(x_0) + \{0\} \subset F(x_o) + F(t) + \overline{V} + K \quad t \in U. \tag{11}$$

Since $F(x_0)$ is a bounded set and $F(t) + \overline{V}$ is a convex set, then by Lemma 2, we have

$$\{0\} \subset \overline{\overline{V} + F(t) + K}$$

for all $t \in U$. Note that the set $\overline{V} + F(t) + K$ is closed as a sum of compact and closed set. Consequently, by condition $F(0) = \{0\}$, we obtain

$$F(0) \subset \overline{V} + F(t) + K \subset F(t) + W + K$$

for all $t \in U$, which means F is K-l.s.c. at zero. □

This article is the introduction to the discussion on the K-continuity problem for K-superquadratic set-valued functions. In the theory of K-subquadratic and K-superquadratic set-valued functions an important role is played by theorems giving possibly weak conditions under which such multifunctions are K-continuous.

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AUTOMATIC SEARCH OF RATIONAL SELF-EQUIVALENCES

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Abstract

Two Witt rings that are not strongly isomorphic (i.e., two Witt rings over two fields that are not Witt equivalent) have different groups of strong automorphisms. Therefore, the description of a group of strong automorphisms is different for almost every Witt ring, which requires the use various tools in proofs. It is natural idea to use computers to generate strong automorphisms of the Witt rings, which is especially effective in the case of the finitely generated Witt rings, where a complete list of strong automorphisms can be created. In this paper we present the algorithm that was used to generate strong automorphisms from the infinite group of strong automorphisms of the Witt ring of rational numbers $W(\mathbb{Q})$.
Keywords: algebra, rational self-equivalences, Witt ring, strong automorphism, algorithm, automatic search

1. Introduction

One of fundamental notions in algebraic number theory of quadratic forms is introduced in [11] ring called nowadays Witt ring. This ring carries information about the behaviour of all quadratic forms over fixed field, hence the structure of Witt ring depends strongly on the field. Two fields are said to be *Witt equivalent* if their Witt rings are isomorphic and considerd isomorpism preserves dimension of quadratic forms (*strong isomorphism*). We consider strong automorphisms of Witt rings and from above inndicate that two non-isomorphic Witt rings have different groups of strong automorphisms. Therefore the investigation of strong automorphisms of Witt rings is a difficult task because of variety of structure of Witt rings. It is a little easier to determine the groups of strong automorphisms of the Witt rings, which are generated by the finite groups of squares classes. In simple

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cases one can list all strong automorphisms by hand. In rings with more complex structures, the natural idea is to use a computer to generate all strong automorphisms. Previous attempts have shown the effectiveness of algorithmic methods in linear algebra (see for example [3]). In literature there are descriptions of groups of strong automorphisms for many wide classes of Witt rings: [4], [5], [7], [8]. Some of the results were verified using computer programs [9].

The search for strong automorphisms is more difficult in the case of Witt rings, which are not finitely generated. The first step in this field may be the result from [1], where it has been shown that the group of strong automorphisms of global fields is uncountable. In this article, we deal with strong automorphisms of the Witt ring $W(\mathbb{Q})$ of the field of rational numbers as a special case of Witt ring of a global field. We present the algorithms used in the computer program that was used in [6] to generate strong automorphisms of the Witt ring $W(\mathbb{Q})$.

2. Algebraic background

In [2] authors showed that two global fields are Witt equivalent (and their Witt ring are strong isomorphic) if and only if they are *Hilbert-symbol equivalennt*. A *Hilbert-symbol equivalence* of two global fields K and L is a pair (T, t), where $T\colon \Omega(K) \to \Omega(L)$ is a bijection between the sets of primes of these fields and $t\colon K^*/K^{*2} \to L^*/L^{*2}$ is an isomorphism of their square class groups which preserves Hilbert symbols with respect to the corresponding primes, i.e.

$$(a, b)_{\mathfrak{p}} = (t(a), t(b))_{T(\mathfrak{p})} \quad \text{for all} \;\; a, b \in K^*/K^{*2},\; \mathfrak{p} \in \Omega(K).$$

The Hilbert symbol equivalence, where $K = L$ is called *Hilbert-symbol self-equivalence* of K.

We conscider the case $K = L = \mathbb{Q}$. Using results from [2] we conclude that for every pair (T, t), which is a Hilbert-symbol self equivalence of the field $\mathbb{Q}$ (called *rational self-equivalence*), the map $\langle a_1, \dots, a_n\rangle \to \langle t(a_1), \dots, t(a_n)\rangle$ induces a strong automoorphism of Witt ring $W(\mathbb{Q})$ of the field of rational numbers. Conversely, every strong automorphism of $W(\mathbb{Q})$ determines uniquely a rational self-equivalence (T, t).

In this case we can deal with prime numbers instead of prime ideals and Hilbert symbols depends only on Legendre symbols ([6], Lemma 2.1). The construction of rational self-equivalences presented in [6] bases on the notion of *small equivalence* introduced in [2]. To make the reading of the next part easier, we will cite some notions and several facts proved in [6].

Let $\mathbb{P}$ denotes the set of prime numbers together with the symbol ∞. For every prime number there is defined a completion $\mathbb{Q}_p$ of the field $\mathbb{Q}$ with the help of valuation v_p called *p-adic number field.* Moreover we agree, that $\mathbb{Q}_\infty = \mathbb{R}$ is a completion of the field $\mathbb{Q}$ at the usual absolute value.

A finite, nonempty set $S \subset \mathbb{P}$ containing 2 and ∞ is called *sufficiently large.* Let S be sufficiently large set of prime numbers $S = \{p_1 \ldots, p_n\}$ and assume that $p_1 = \infty$, $p_2 = 2$. The *set of S-singular elements* is defined as follows:

$$E_S = \{x \in \mathbb{Q}^*\colon\ v_p(x) \equiv 0 \pmod 2 \ \text{ for all } \ p \notin S\}.$$

Notice that E_S is a subgroup of the multiplicative group of the field $\mathbb{Q}$ containing all squares of rational numbers. Therefore the quotient group $E_S/\mathbb{Q}^{*2}$ is a subgroup of the group $\mathbb{Q}^*/\mathbb{Q}^{*2}$. By the definition of the set E_S every element $x \in \mathbb{Q}$ has the factorization

$$x = (-1)^{e_1} 2^{2k_2+e_2} p_3^{2k_3+e_3} \cdots p_n^{2k_n 3+e_n} q_1^{2l_1} \cdots q_m^{2l_m},$$

where $q_1, q_2, \ldots, q_m \notin S$ are prime numbers, $k_i, l_i \in \mathbb{Z}$ and $e_i \in \{0, 1\}$. Then

$$x\mathbb{Q}^{*2} = (-1)^{e_1} 2^{e_2} p_3^{e_3} \cdots p_n^{e_n} \mathbb{Q}^{*2}.$$

It follows that the elements of the group $E_S/\mathbb{Q}^{*2}$ are represented by the integers of the form $(-1)^{e_1} 2^{e_2} p_3^{e_3} \cdots p_n^{e_n}$ in the unique way.

For every $p \in \mathbb{P}$ the natural imbedding of the field $\mathbb{Q}$ in the field $\mathbb{Q}_p$ induces the group homomorphism $i_p\colon \mathbb{Q}^*/\mathbb{Q}^{*2} \to \mathbb{Q}_p^*/\mathbb{Q}_p^{*2}$, which is surjective. For the finite set $S = \{p_1, \ldots, p_n\} \subset \mathbb{P}$ we get the dual homomorphism $\mathrm{diag}_S\colon \mathbb{Q}^*/\mathbb{Q}^{*2} \to \prod_{p\in S} \mathbb{Q}_p^*/\mathbb{Q}_p^{*2}$ defined by

$$\mathrm{diag}_S(a) = [i_{p_1}(a), \ldots, i_{p_n}(a)] = [a\mathbb{Q}_{p_1}^{*2}, \ldots, a\mathbb{Q}_{p_n}^{*2}].$$

Definition 1. *Let S be sufficiently large set of prime numbers defined as above. A* small S-equivalence *is a pair $\mathcal{R} = ((t_p)_{p\in S}, T)$, where*
1) $T\colon S \to T(S)$ is a bijection,
2) there exists the isomorphism of the group of square classes $t_S : E_S/\mathbb{Q}^{*2} \to E_{T(S)}/\mathbb{Q}^{*2}$,
3) $(t_p)_{p\in S}$ is a family of local isomorphisms $t_p : \mathbb{Q}_p^*/\mathbb{Q}_p^{*2} \to \mathbb{Q}_{T(p)}^*/\mathbb{Q}_{T(p)}^{*2}$ preserving Hilbert symbols, i.e.

$$(a, b)_p = (t_p(a), t_p(b))_{T(p)} \quad \textit{for all} \quad a, b \in \mathbb{Q}_p^*/\mathbb{Q}_p^{*2},$$

4) the following diagram commutes

$$\begin{array}{ccc}
E_S/\mathbb{Q}^{*2} & \xrightarrow{i_S} & \prod_{p\in S} \mathbb{Q}_p^*/\mathbb{Q}_p^{*2} \\
\downarrow t_S & & \downarrow \prod t_p \\
E_{T(S)}/\mathbb{Q}^{*2} & \xrightarrow{i_{T(S)}} & \prod_{p\in S} \mathbb{Q}_{T(p)}^*/\mathbb{Q}_{T(p)}^{*2}
\end{array}$$

It was shown in [6] that any small S_k-equivalence $\mathcal{R}_{S_k} = ((t_p)_{p\in S_k}, T)$, where $S_k = \{\infty, 2, p_3, p_4, \ldots, p_k\}$ is sufficiently large set of prime numbers can be extended to some small S'_{k+1}-equivalence, where $S'_{k+1} : = S'_k \cup \{q_{k+1}\}$ and there is infinitely many prime numbers, which can be choosen as q_{k+1} provided they fulfill the following two (sufficent) condoitions:
1) $p_{k+1} \equiv q_{k+1} \pmod 8$,
2) $\left(\frac{p_i}{p_{k+1}}\right) = \left(\frac{q_i}{q_{k+1}}\right)$ for all $3 \le i \le k$
and the last Hilbert symbols depend only on Legendre symbols. Above conditions ensure comutativity of suitable diagrams (cf. [6]). In conclusion any small equivalence can be extended to some rational self-equivalence, which induces strong automorphism of Witt ring $W(\mathbb{Q})$ of the field of rational numbers.

3. Algorithm for building of sufficiently large sets

The computer program that performs the search of rational self-equiva-len-ces consists of several stages and must be stopped at some point (because it is not possible to generate prime numbers infinitely).

We start from the sufficiently large sets $S' = SP' = \{\infty, 2\}$. Let us first remark that the definition of small equivalence imposes some restrictions on the mapping of T. Namely $T(\infty) = \infty$ and $T(2) = 2$. Then we take the smallest prime number $p_3 \notin S'$, i.e. $p_3 = 3$ and now we get expanded set $S' = \{\infty, 2, 3\}$. Then we search for prime number q_3 which is outside of SP' and fulfills $p_3 \equiv q_3 \pmod 8$. It turns out to be prime number 11. Let us denote this step of construction in the following way:
1) $p_3 = 3 \to 11 = q_3$.
(Notice, that we have to assume that $p_3 \neq q_3$. If we take $p_3 = q_3$ and continue in this way, we get identity).

Next we take the smallest prime number q_4, which was not used in the sequence $SP' = \{q_i\}_{i=1}^{\infty}$. It is number 3. We search for $q_4 = 3$ the smallest prime number p_4, which has required properties:
i) $q_4 \equiv p_4 \pmod 8$ and
ii) $\left(\frac{q_3}{q_4}\right) = \left(\frac{p_3}{p_4}\right)$.
It is the number 19. Hence we denote the second step of the construction:
2) $p_4 = 19 \leftarrow 3 = q_4$.
Further steps of construction lead to the following sequences of prime numbers
$S : 3, 19, 5, 13, 7, 1103, 11, 6329, 17, 347, 23, 77551, 29, 138581, 31,$
$SP : 11, 3, 13, 5, 223, 7, 283, 17, 2689, 19, 31159, 23, 109229, 29, 1010903,$
what gives the following sufficiently large sets:
$S' = \{\infty, 2, 3, 19, 5, 13, 7, 1103, 11, 6329, 17, 347, 23, 77551, 29, 138581, 31\},$

$SP' = \{\infty, 2, 3, 11, 3, 13, 5, 223, 7, 283, 17, 2689, 19, 31159, 23, 109229, 29, 1010903\}$
and the map T:
$T(\infty) = \infty$,
$T(2) = 2$,
$T(3) = 11$,
$T(19) = 3$,
$T(5) = 13$,
$T(13) = 5$,
$T(7) = 223$,
$T(1103) = 7$,
$T(11) = 283$,
$T(6329) = 17$,
$T(17) = 2689$,
$T(347) = 19$,
$T(23) = 31159$,
$T(77551) = 23$,
$T(29) = 109229$,
$T(138581) = 29$,
$T(31) = 1010903$
which easily shows how the next small equivalences are constructed. The limitation to 15 steps is due to the rapid increase of searched prime numbers. This process, continued into infinity, gives us a rational self-equivalence.

Of course the choice of another q_3 gives another sequences of prime numbers p_k and q_k and the different sequences of small equivalences (for another examples of rational self-equivalences searched in this way see [6]).

Now we show how we construct two sequences S and SP of prime numbers using Algorithm 1.

Algorithm **built_ sequences**() inputs the set P of prime numbers generated by sieve of Eratosthenes. It uses the function $FindElement()$ as defined in Algorithm 2. First we add 3 to the set S as a smallest prime number (line 2). The variable p is initilized as 3 (line 4). (The variable p and q are used to build the sets S and SP, respectively.) q is initialized as 0 (line 3). As long as the variable i is less than 15 the algorithm performs the following: for odd runs it searches a smallest prime number q by using function $FindElement()$ (line 7) and adds it to the set SP; next finds the first free number prime by using function $FirstFree()$ (line 9); gets it to q and adds it to the set SP; for even runs the algorithm performs steps described above for the variable p and the set S. Algorithm **built_ sequences**() terminates when i is greater then 15 and returns two sets S and SP.

The function $FindElement()$ inputs the set P of prime numbers, the sets S and SP, the element el and the variable i. It uses the function

---

Algorithm 1: function **built_ sequences**(P)

Variables:

S (sequence of prime numbers, initialized as $\emptyset$)

SP (sequence of prime numbers, initialized as $\emptyset$

i, q, p (integer)

Returned values:

SP, S /* two sequences of prime numbers*/

$i \leftarrow 1$
$S \leftarrow S \cup \{3\}$
$q \leftarrow 0$
$p \leftarrow 3$
while $i \leqslant 15$ **do**
    if $i \; mod \; 2 == 1$ **then**
        $q \leftarrow FindElement(P, S, SP, p, i)$
        $SP \leftarrow SP \cup \{q\}$
        $q \leftarrow FirstFree(P, SP)$
        $SP \leftarrow SP \cup \{q\}$
    else
        $p \leftarrow FindElement(P, S, SP, q, i)$
        $S \leftarrow S \cup \{p\}$
        $p \leftarrow FirstFree(P, S)$
        $S \leftarrow S \cup \{p\}$
    end if
    $i \leftarrow i + 1$
end while
return S,SP

$Legrende()$ defined as one of standard algorithm calculated of Legendre symbol [10]. Algorithm searches for the prime number j (line 3) such that $j \neq el$ AND $(j - el) \; mod \; 8 == 0$ (line 4). Algorithm terminates and returns j when for j and el and sets S and SP all Legrende symbols $Left$ and $Right$ are equal (lines 5-17), respectively.

The algorithms were implemented in C++. The experiments were carried out on an notebook Intel Core i5-5200U CPU 2.20 GHz, 8 GB RAM with Linux operation system.

4. Final remarks

In this case the obtained results have shown the usefulness of the computer. The value of the greatest searched prime number in the example described in previous section shows that it would be extremely time-consuming

Algorithm 2: function **FindElement**(P,S,SP,el,i)

Variables:

$result$ (boolean variable, initialized as **False**)

j (integer)

Returned values:

j /* prime number*/

```
result ← False
while NOT result do
  if i mod 2 == 1 then
    j ← NextPrime(P, SP)
  else
    j ← NextPrime(P, S)
  end if
  if j ≠ el AND (j − el) mod 8 == 0 then
    result ← True
    k ← 1
    while k < i AND result do
      if i mod 2 == 1 then
        Left ← Legendre(S[k], el)
        Right ← Legendre(SP[k], j)
      else
        Left ← Legendre(S[k], j)
        Right ← Legendre(SP[k], el)
      end if
      result ← Left == Right
      k ← k + 1
    end while
  end if
end while
return j
```

or impossible at all to do the calculations without a computer. This allows usu to think that the computer would be useful in solving similar problems in the future.

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